CUTS IN THE ML DEGREES

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Abstract. We show that the cut defined by a real number $r \in [0, 1]$ is realised in the hierarchy of p-bases in the ML degrees if and only if it is left-Π^0_3.

1. Introduction

In [1], the authors characterise the sets that are computable from some pair of relatively random sequences, or equivalently, from both halves of some ML-random sequence. There are only countably many such sets, they are all K-trivial, and the Turing degrees of these sets form an ideal. It turns out that this ideal is one among a hierarchy of ideals $\mathcal{B}_p$ in the K-trivial degrees, each indexed by rationals $p \in [0, 1]$, with $p < q$ implying that $\mathcal{B}_p \subsetneq \mathcal{B}_q$. If $p = k/n$ with $k < n$ natural numbers, then $\mathcal{B}_p$ is the collection of sets A which for some random sequence Z (equivalently, for $Z = \Omega$ being any left-c.e. random sequence), A is computable from the join of any k of the n-columns of Z. Various similar characterizations of these ideals are known; for example, see [1, Prop. 5.1].

Since the $\mathcal{B}_p$ are a strictly ordered chain of ideals, it is natural to ask: which cuts are realised? Namely for which reals $r \in (0, 1)$ is there a set A that is an element of $\mathcal{B}_p$ exactly for $p > r$? There are only countably many K-trivial sets, and so only countably many cuts are realised this way. In this paper we characterise these cuts:

Theorem 1.1. *The following are equivalent for a real number $r \in (0, 1)$:*

(1) *There is a set A such that for all $p \in \mathbb{Q} \cap [0, 1]$, $A \in \mathcal{B}_p \iff p > r$.*
(2) *r is right-Σ^0_3.*

By (2), we mean that the right cut $\{p \in \mathbb{Q} : p > r\}$ is Σ^0_3. We note that since each ideal $\mathcal{B}_p$ is characterised by being computable from a collection of random sequences, [2, Thm. 2.1] implies that we may take A to be c.e. in (1).

Remark 1.2. When $r \in (0, 1)$ is rational, the conditions of Theorem 1.1 hold. However, in this case, one can also ask whether there is a set A with $A \in \mathcal{B}_p \iff p \geqslant r$. A positive answer follows from [2, Thm. 3.3]. Alternatively, the construction below can be modified to obtain such a set A.

The main tool used to explore the ideals $\mathcal{B}_p$ is *cost functions*. We recall some definitions. A *cost function* is a computable function $\mathbf{c} \colon \mathbb{N}^2 \to \mathbb{R}^{\geqslant 0}$. In this paper we only consider cost functions $\mathbf{c}$ with the following extra properties:

(i) *Monotonicity*: for all x and s, $\mathbf{c}(x, s) \leqslant \mathbf{c}(x, s+1)$ and $\mathbf{c}(x, s) \geqslant \mathbf{c}(x+1, s)$;
(ii) The *limit condition*: for all x, $\underline{\mathbf{c}}(x) = \lim_s \mathbf{c}(x, s)$ is finite and $\lim_{x \to \infty} \underline{\mathbf{c}}(x) = 0$;

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(iii) For all x and s, $\mathbf{c}(x,s) \leqslant 1$;
(iv) For all $s < x$, $\mathbf{c}(x,s) = 0$.

The idea is that a cost function $\mathbf{c}$ measures, in an analytic way, the complexity of a computable approximation $\langle A_s \rangle$ of a Δ^0_2 set A. Intuitively, the fewer the mind-changes, the simpler A is. The number $\mathbf{c}(x,s)$ is the cost of changing A on x at stage s, namely of setting $A_s(x) \neq A_{s-1}(x)$. The monotonicity condition says that the cost of changing x goes up as time passes, and that at any given stage, it is cheaper to change A on larger numbers. The limit condition puts a restraint on the costs, ensuring they are not too onerous in the limit. The notion of obedience tells us which computable approximations are simple from $\mathbf{c}$'s point of view:

Definition 1.3. Let $\langle A_s \rangle$ be a computable approximation of a Δ^0_2 set A, and let $\mathbf{c}$ be a cost function. The *total* $\mathbf{c}$*-cost* of $\langle A_s \rangle$ is

$$\mathbf{c}\langle A_s \rangle = \sum_{s<\omega} \mathbf{c}_s(x) \, [\![x \text{ is least such that } A_s(x) \neq A_{s-1}(x)]\!].$$

We say that A *obeys* $\mathbf{c}$ if for some computable approximation $\langle A_s \rangle$ of A, $\mathbf{c}\langle A_s \rangle$ is finite.

In [1], it is shown that for all rational $p \in (0,1)$, $A \in \mathcal{B}_p$ if and only if A obeys the cost function $\mathbf{c}_{\Omega,p}$ defined by

$$\mathbf{c}_{\Omega,p}(x,s) = \begin{cases} (\Omega_s - \Omega_x)^p, & \text{if } x \geqslant s; \\ 0, & \text{if } x < s. \end{cases}$$

Here $\langle \Omega_s \rangle$ is some increasing computable approximation of a left-c.e. ML-random sequence Ω. This characterisation of the ideals $\mathcal{B}_p$ shows that Theorem 1.1 is really a theorem about cost functions. For two cost functions $\mathbf{c}$ and $\mathbf{c}'$, write $\mathbf{c} \ll \mathbf{c}'$ if:

- for all x and s, $\mathbf{c}(x,s) \leqslant \mathbf{c}'(x,s)$; and
- for every constant k, $\underline{\mathbf{c}}'(x) > k\underline{\mathbf{c}}(x)$ for all but finitely many x.

We prove:

Proposition 1.4. *Let $\{\mathbf{c}_p : p \in \mathbb{Q} \times (0,1)\}$ be a collection of uniformly computable cost functions, such that if $p < q$, then $\mathbf{c}_q \ll \mathbf{c}_p$. Then for any real number $r \in (0,1)$, the following are equivalent:*

(1) *There is a set A such that for all $p \in \mathbb{Q} \cap [0,1]$, A obeys $\mathbf{c}_p$ if and only if $p > r$.*
(2) *r is right-Σ^0_3.*

It is readily observed that $\mathbf{c}_{\Omega,q} \ll \mathbf{c}_{\Omega,p}$ whenever $p < q$, and so Proposition 1.4 implies Theorem 1.1.

2. Proof of Proposition 1.4

Before we prove Proposition 1.4, we introduce some notation and state a lemma. Suppose that $\langle A_s \rangle$ is a computable approximation of a set A. A *speed-up* of $\langle A_s \rangle$ is an approximation $\langle A_{h(s)} \rangle$ where $h \colon \mathbb{N} \to \mathbb{N}$ is computable and strictly increasing. For simplicity, we write $\langle A_h \rangle$ for $\langle A_{h(s)} \rangle$. It is not difficult to see that if $\langle A_h \rangle$ is a speed-up of $\langle A_s \rangle$, then for any cost function $\mathbf{c}$, $\mathbf{c}\langle A_h \rangle \leqslant \mathbf{c}\langle A_s \rangle$. In fact, there are several reasons that the cost on the left might be smaller. Suppose that x is the least such that $A_{h(s)}(x) \neq A_{h(s-1)}(x)$. So the step s contribution to $\mathbf{c}\langle A_h \rangle$ is $\mathbf{c}(x,s)$. In contrast, the step $h(s)$ contribution to $\mathbf{c}\langle A_s \rangle$ is at least $\mathbf{c}(x,h(s))$, which by monotonicity is at least $\mathbf{c}(x,s)$. It may be more, since it is possible that there is some $y < x$ such that $A_{h(s)}(y) \neq A_{h(s)-1}(y)$, but it just happens that $A_{h(s)}(y) = A_{h(s-1)}(y)$. And of course, relative to $\langle A_s\rangle$, $\mathbf{c}\langle A_h\rangle$ only counts some of the stages, namely those in the range of h. We will make use of the following, which is well-known, and follows from the techniques in [3]:

Lemma 2.1. A Δ^0_2 set A obeys a cost function $\mathbf{c}$ if and only if every computable approximation $\langle A_s\rangle$ of A has a speed-up $\langle A_h\rangle$ with $\mathbf{c}\langle A_h\rangle < \infty$.

We fix an effective listing $\langle h_e\rangle$ of partial "speed-up" functions. That is:

- $\langle h_e\rangle$ are uniformly partial computable;
- Each h_e is either total, or its domain is a finite initial segment of ω;
- Each h_e is strictly increasing on its domain;
- Every strictly increasing computable function is h_e for some e.

Further, for every e and s, let $n_{e,s} = \max \operatorname{dom} h_{e,s}$; by withholding convergences, we may assume that:

- $\operatorname{dom} h_{e,s}$ is an initial segment of ω; and
- $h_{e,s}(n_{e,s}) < s$.

For any cost function $\mathbf{c}$ we can define

$$\mathbf{c}\langle A_{h_e}\rangle[s] = \sum_{m\leqslant n_{e,s}} \mathbf{c}(x,m)\ [\![x \text{ is least such that } A_{h_e(m)} \neq A_{h_e(m-1)}]\!].$$

The value $\mathbf{c}\langle A_{h_e}\rangle[s]$ is computable, uniformly in e, s and in a computable index for $\mathbf{c}$. And if h_e is total, then $\mathbf{c}\langle A_{h_e}\rangle = \lim_s \mathbf{c}\langle A_{h_e}\rangle[s]$.

(1) $\Longrightarrow$ (2) of Proposition 1.4 is essentially [3, Fact 2.13], which is uniform. We are given a Δ^0_2 set A; we fix a computable approximation $\langle A_s\rangle$ for A. By Lemma 2.1, A obeys $\mathbf{c}_p$ if and only if there are some e and M such that h_e is total and for all s, $\mathbf{c}_p\langle A_{h_e}\rangle[s] \leqslant M$. This is a Σ^0_3 predicate of p. Note that the collection of p such that A obeys $\mathbf{c}_p$ must be a right cut (a final segment of $\mathbb{Q} \cap (0,1)$); this follows from the assumption that $\mathbf{c}_q \leqslant \mathbf{c}_p$ for $p < q$.

Before we give the details, we briefly discuss the proof of (2) $\Longrightarrow$ (1). We are given a right-Σ^0_3 real $r \in (0,1)$, and define a computable approximation $\langle A_s\rangle$ of the desired set A. The value of r can be guessed by the true path on a tree of strategies: one duty of the strategies is to guess, given $p \in \mathbb{Q} \cap (0,1)$, whether $p > r$ or not; locally the behaviour of the true path is Σ^0_2/Π^0_2, so to approximate the Σ^0_3 predicate $p > r$, we need to keep trying different existential witnesses for the outermost quantifier.

Suppose that a strategy τ works with some rational number $p = p^\tau$. There are two possibilities. The infinite outcome $\tau\hat{}\infty$ believes that it has proof that $p > r$, and so it is $\tau\hat{}\infty$'s responsibility to ensure that A obeys $\mathbf{c}_p$. This is both done passively, by initialisations, and more actively, by setting strict bounds on the action of weaker requirements. The speed-up of $\langle A_s\rangle$ which witnesses that A obeys $\mathbf{c}_p$ is the restriction of our approximation to the $\tau\hat{}\infty$-stages. There are two kinds of nodes σ that my change A, and thus increase the cost measured by $\tau\hat{}\infty$: nodes to the right of $\tau\hat{}\infty$, and nodes extending $\tau\hat{}\infty$. For each node σ we assign a bound δ^σ on the amount of cost that σ's action may cause to nodes (strategies) strictly above it (nodes that σ extends). We distribute the bounds δ^σ so that the total damage caused by all nodes extending $\tau\hat{}\infty$ is finite. The nodes to the right of $\tau\hat{}\infty$ (including the finite outcome $\tau\hat{}\mathtt{fin}$) contribute *nothing* to $\tau\hat{}\infty$'s cost. This is the result of initialisations and our speed-up: at the m^{th} $\tau\hat{}\infty$ stage, nodes to the right only change A on numbers greater than m, and we measure the $\mathbf{c}_p$-cost of these changes at stage m. We use the assumption (iii) above, that if $s < x$ then $\mathbf{c}_p(x, s) = 0$.

Now consider the Σ^0_2 outcome $\tau\hat{}\mathtt{fin}$. This outcome believes that $p \leqslant r$, and so tries to ensure that A does not obey $\mathbf{c}_p$. By Lemma 2.1, it suffices to check all speed-ups of our base approximation $\langle A_s\rangle$. We make use of the following strengthening of Lemma 2.1:

Lemma 2.2 (Fact 2.2 of [3])**.** Suppose that $\langle A_s\rangle$ is a computable approximation of a set A that obeys a cost function $\mathbf{c}$. Then for any $\varepsilon > 0$, there is a speed-up of $\langle A_s\rangle$ with total cost bounded by ε.

Thus, in order to show that A does not obey $\mathbf{c}_p$, it suffices to ensure that for all e, $\mathbf{c}_p\langle A_{h_e}\rangle \geqslant 1$. The node τ will be assigned one e. It needs to change A on numbers x so that the cost $\mathbf{c}_p\langle A_{h_e}\rangle$ increases. The node τ faces two difficulties:

- Some nodes above τ restrain τ from adding more than δ^τ to their cost; and δ^τ is much smaller than 1.
- The speed-up function h_e is revealed to τ very slowly.

The second difficulty is technical: we see $h_e(m)$ converge to some value t only at some stage s much later than t. Thus, τ discovers that it had to change A_t on some value; but A_t was already defined at stage t. This is addressed easily by giving τ an infinite collection (which we denote by $\omega^{[\tau]}$) of potential inputs for x to play with; for a suitable $x \in \omega^{[\tau]}$, the node τ keeps $A_r(x) \neq A_t(x)$ for stages $r \geqslant s$ until we see a value of h_e greater than s.

The first difficulty is fundamental: this is where we use the assumptions on the relative growth-rate of the cost functions $\mathbf{c}_p$. Take some node τ working to increase $\mathbf{c}_q\langle A_{h_e}\rangle$ for some e and q, and let ρ be some node above τ that is concerned about incurring cost from τ's action. The node ρ only cares if it is trying to keep costs low; that is, if $\rho\hat{}\infty \leqslant \tau$. Let $p = p^\rho$ be the rational number that ρ is working with; it is trying to keep the $\mathbf{c}_p$-cost of some approximation finite. Now the outcome $\rho\hat{}\infty$, and therefore τ, believe that they have proof that $p > r$. The node τ is working with the assumption that $q \leqslant r$. Thus, we can arrange that $q < p$. The assumption $\mathbf{c}_q \ll \mathbf{c}_p$ now means that τ can change A to make the $\mathbf{c}_q$-cost large while keeping the $\mathbf{c}_p$-damage very small: smaller than δ^τ.

We now give the details. Let $r \in (0,1)$ be right-Σ^0_3. There are uniformly computable, non-decreasing sequences $\langle \ell^{p,e}_s\rangle_{s<\omega}$ (of natural numbers) for $p \in \mathbb{Q}$ and $e < \omega$ such that for all such p, $p > r$ if and only if for some $e < \omega$, $\langle \ell^{p,e}_s\rangle$ is unbounded.

We define a computable approximation $\langle A_s\rangle$ of a Δ^0_2 set A. We will meet two types of requirements. The first type of requirements are indexed by $p \in \mathbb{Q} \cap (0,1)$:

$$N_p : \text{If } p > r, \text{ then } A \text{ obeys } \mathbf{c}_p.$$

Requirements of the second type are indexed by $p \in \mathbb{Q} \cap (0,1)$ and $e < \omega$:

$$R_{p,e} : \text{If } p < r \text{ and } h_e \text{ is total, then } \mathbf{c}_p\langle A_{h_e}\rangle \geqslant 1.$$

As discussed above, meeting these requirements suffices to ensure (1) of the proposition.

Approximating r. We work with a full binary tree of strategies. The strategies are the finite sequences of the symbols ∞ and $\mathtt{fin}$.

By recursion on the length $|\sigma|$ of a node σ on the tree, we define:

- $p^\sigma \in \mathbb{Q} \cap (0,1)$ and $e^\sigma \in \omega$; the node σ will attempt to meet either N_{p^σ} or R_{p^σ, e^σ};
- a rational number $r^\sigma > p^\sigma$; this is an upper bound on the value of r believed by σ.

The meaning of the outcome ∞ is that we believe that $p^\sigma > r$, and so we meet N_{p^σ} by defining a suitable speed-up of our approximation for A. The meaning of the outcome $\mathtt{fin}$ is that we believe that $p^\sigma \leqslant r$, and so we meet R_{p^σ, e^σ}.

We use an effective ω-ordering of all the pairs $(p, e) \in (\mathbb{Q} \cap (0,1)) \times \omega$. We start with the root of the tree, which is the empty sequence $\langle\rangle$, by letting $(p^{\langle\rangle}, e^{\langle\rangle})$ be the least pair in our ordering; we let $r^{\langle\rangle} = 1$.

Suppose that σ is on the tree and that we have already defined p^σ, e^σ and r^σ. We then define these parameters for the children $\sigma\hat{}\infty$ and $\sigma\hat{}\mathtt{fin}$. We start with the latter:

(a) $r^{\sigma\hat{}\infty} = p^\sigma$.
(b) $r^{\sigma\hat{}\mathtt{fin}} = r^\sigma$.

Then, for both children τ of σ, we let (p^τ, e^τ) be the next pair (p, e) on our list after (p^σ, e^σ) such that $p < r^\tau$.

For brevity, for any node σ, we write:

- ℓ_s^σ for $\ell_s^{p^\sigma, e^\sigma}$.
- h^σ for h_{e^σ} (and similarly h_s^σ for $h_{e^\sigma, s}$).

Allocating capital to nodes. Computably, we assign to each node σ a positive rational number δ^σ such that

$$\sum \delta^\sigma \leqslant 1$$

(where the sum ranges over all strategies σ). The idea of the parameter δ^σ is that σ promises any τ with $\tau\hat{}\infty \leqslant \sigma$ that it will not add more than δ^σ to the cost accrued by τ.[1]

Construction. At stage s, we define the path of accessible nodes by recursion. If a strategy σ is accessible at stage s, then we say that s is a *σ-stage*.

We start with $A_0 = 0^\infty$.

The root is always accessible. Suppose that a node σ is accessible at stage s. If $|\sigma| = s$, we halt the stage. We also initialise all nodes weaker than σ.

Suppose that $|\sigma| < s$.

First, let $t < s$ be the last $\sigma\hat{}\infty$-stage before stage s; $t = 0$ if there was no such stage. If $\ell_s^\sigma > t$, then we let $\sigma\hat{}\infty$ be the next accessible node.

Suppose that $\ell_s^\sigma \leqslant t$. We will define the notion of a *σ-action* stage. Let w be the last σ-action stage prior to stage s; $w = 0$ if there was no such stage. Let

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[1] Actually, it will be $2\delta^\sigma$, for a truly unimportant reason. The last σ-action may add to the cost σ is measuring a quantity close to 1, making the total cost close to 2; from τ's point of view, the increase is then close to $2\delta^\sigma$.

More importantly, note that the value δ^σ does not depend on the stage number. A reasonable approach would be to shrink δ^σ each time σ is initialised. We do not need to do this, because even when σ is initialised, the amount that it previously added to the total cost it is monitoring has not gone away, and so it does not need to start afresh.

$n = \max \operatorname{dom} h^\sigma_s$; let s^* be the last stage prior to stage s at which $\sigma\hat{}\mathtt{fin}$ was initialised. If:

(i) $\mathbf{c}_{p^\sigma}(\langle A_{h^\sigma}\rangle)[s] < 1$;
(ii) $n > w$; and
(iii) there is a number $x > s^*$, $x \in \omega^{[\sigma]}$ satisfying[2]

$$\mathbf{c}_{r^\sigma}(x, s) \leqslant \mathbf{c}_{p^\sigma}(x, n) \cdot \delta^\sigma,$$

then we choose the least such x, set $A_{s+1}(x) = 1 - A_s(x)$, and call s a σ-action stage. Otherwise, σ makes no change to A at stage s. In either case, we let $\sigma\hat{}\mathtt{fin}$ be the next accessible node.

2.1. **Verification.** Let δ^* denote the true path. Because we never terminate a stage s before we get to a node of length s, and the strategy tree is binary splitting, the true path is infinite.

Toward verifying that the requirements are met, we show that the true path approximates r correctly. For the first part of the next lemma, note that if τ extends σ, then $r^\sigma \geqslant r^\tau$, so $\inf_{\sigma\in\delta^*} r^\sigma = \lim_{\sigma\in\delta^*} r^\sigma$.

Lemma 2.3.

(a) $r = \inf_{\sigma\in\delta^*} r^\sigma$.
(b) For all rational $p \in (0, r)$, for all e, there is some $\sigma \in \delta^*$ with $(p^\sigma, e^\sigma) = (p, e)$.

Proof. First, by induction on the length of $\sigma \in \delta^*$ we verify that $r^\sigma > r$. For the root this is clear since $r < 1$. If $\sigma \in \delta^*$ and $r^\sigma > r$, there are two cases. If $\sigma\hat{}\infty \in \delta^*$ then $\langle \ell^\sigma_s\rangle$ is unbounded, which implies that $p^\sigma = r^{\sigma\hat{}\infty} > r$. Otherwise $\sigma\hat{}\mathtt{fin} \in \delta^*$ and $r^{\sigma\hat{}\mathtt{fin}} = r^\sigma > r$.

Let $\tilde{r} = \inf_{\sigma\in\delta^*} r^\sigma$. Let $p \in (0, \tilde{r})$ be rational and let $e < \omega$. For all $\tau \in \delta^*$, $r^\tau > p$. Thus, we never skip over the pair (p, e) when assigning pairs to the nodes on the true path. It follows that there is some $\sigma \in \delta^*$ with $(p^\sigma, e^\sigma) = (p, e)$. This verifies (b).

Suppose, for a contradiction, that $\tilde{r} > r$. Let $p \in (r, \tilde{r})$ be rational, and let e witness that $p > r$, that is, $\langle \ell^{p,e}_s\rangle$ is unbounded. Let $\sigma \in \delta^*$ with $(p^\sigma, e^\sigma) = (p, e)$. Then $\sigma\hat{}\infty \in \delta^*$ and $r^{\sigma\hat{}\infty} = p < \tilde{r}$, which is a contradiction. □

The next lemma shows that action by a node does increase the total cost it is monitoring. Let σ be any node, and let s be a σ-action stage. We let

- $n^\sigma_s = \max \operatorname{dom} h^\sigma_s$;
- y^σ_s be the number acted upon by σ, that is, the unique number $y \in \omega^{[\sigma]}$ such that $A_{s+1}(y) \neq A_s(y)$.

Lemma 2.4. Let σ be any node, and suppose that $s < s'$ are two σ-action stages. Then

$$\mathbf{c}_{p^\sigma}(\langle A_{h^\sigma}\rangle)[s'] \geqslant \mathbf{c}_{p^\sigma}(\langle A_{h^\sigma}\rangle)[s] + \mathbf{c}_{p^\sigma}(y^\sigma_s, n^\sigma_s).$$

Proof. Let $y = y^\sigma_s$. We may assume that $s' = s^+$ is the next σ-action stage after stage s. Also let s^- be the previous σ-action stage prior to stage s ($s^- = 0$ if there was no such stage). Since σ does not act between stages s^- and s, and between stages s and s^+,

- $A_t(y)$ is constant for $t \in (s^-, s]$; and

[2]Recall that $\omega^{[\rho]}$, for $\rho \in \{\infty, \mathtt{fin}\}^{<\omega}$, is a partition of ω into pairwise disjoint, infinite computable sets.

- $A_t(y)$ is constant for $t \in (s, s^+]$.

The point is that no other node can change A on an element of $\omega^{[\sigma]}$. Note that

$$s^- < n^\sigma_s \leqslant h^\sigma(n^\sigma_s) < s < n^\sigma_{s+} \leqslant h^\sigma(n^\sigma_{s+}) < s^+.$$

Thus there is some $m \in (n^\sigma_s, n^\sigma_{s+}]$ such that $s^- < h^\sigma(m-1) \leqslant s < h^\sigma(m) \leqslant s^+$. Then $A_{h^\sigma(m-1)}(y) \neq A_{h^\sigma(m)}(y)$. This shows that stage m of the approximation A_{h^σ} contributes at least $\mathbf{c}_{p^\sigma}(y, m) \geqslant \mathbf{c}_{p^\sigma}(y, n^\sigma_s)$ to $\mathbf{c}_{p^\sigma}(\langle A_{h^\sigma}\rangle)[s^+]$, and this was not seen at stage s. □

Lemma 2.5. Let τ be any node. Then

$$\sum \{\mathbf{c}_{p^\tau}(y^\tau_s, n^\tau_s) \,:\, s \text{ is a } \tau\text{-action stage}\} < 2.$$

Proof. For $t \leqslant \omega$, let

$$S^\tau_t = \sum \{\mathbf{c}_{p^\tau}(y^\tau_s, n^\tau_s) \,:\, s \text{ is a } \tau\text{-action stage } \;\&\; s < t\}.$$

Then Lemma 2.4 implies that for every τ-action stage s,

$$S^\tau_s \leqslant \mathbf{c}_{p^\tau}(\langle A_{h^\tau}\rangle)[s] < 1.$$

If there are infinitely many τ-action stages then $S^\tau_\omega \leqslant 1$. Otherwise, let s be the last τ-action stage. As $\mathbf{c}_{p^\tau}(y^\tau_s, n^\tau_s) \leqslant 1$, we have

$$S^\tau_\omega = S^\tau_s + \mathbf{c}_{p^\tau}(y^\tau_s, n^\tau_s) < 2. \qquad \square$$

Lemma 2.6. Let σ be a node and suppose that $\sigma\hat{\ }\mathtt{fin} \in \delta^*$. Then there are only finitely many σ-action stages. If h^σ is total then $\mathbf{c}_{p^\sigma}\langle A_{h^\sigma}\rangle \geqslant 1$.

Proof. If h^σ is partial, then there cannot be more than one σ-action stage after stage $\max \operatorname{dom} h^\sigma$. Suppose that h^σ is total. We will show that eventually, $\mathbf{c}_{p^\sigma}(\langle A_{h^\sigma}\rangle)[s] \geqslant 1$, which will also imply that there are only finitely many σ-action stages. Suppose, for a contradiction, that for all s, $\mathbf{c}_{p^\sigma}(\langle A_{h^\sigma}\rangle)[s] < 1$.

Let s^* be the last stage at which $\sigma\hat{\ }\mathtt{fin}$ is initialised. Since $r^\sigma > p^\sigma$, we know that for all but finitely many x,

$$\underline{\mathbf{c}}_{r^\sigma}(x) < \underline{\mathbf{c}}_{p^\sigma}(x) \cdot \delta^\sigma.$$

Let x^* be the least $x > s^*$, $x \in \omega^{[\sigma]}$ satisfying this inequality. Then for all but finitely many stages t, for all s,

$$\mathbf{c}_{r^\sigma}(x^*, s) < \mathbf{c}_{p^\sigma}(x^*, t) \cdot \delta^\sigma.$$

For sufficiently late stages s, we have $n = \max \operatorname{dom} h^\sigma_s > x^*$ and $\mathbf{c}_{r^\sigma}(x^*, s) < \mathbf{c}_{p^\sigma}(x^*, n) \cdot \delta^\sigma$. This shows that there are infinitely many σ-action stages. Let t^* be a late σ-action stage; let $\varepsilon^* = \mathbf{c}_{p^\sigma}(x^*, t^*)$, which is positive. For every σ-action stage $s > t^*$, by minimality of y^σ_s, we have $y^\sigma_s \leqslant x^*$, and as $n^\sigma_s > t^*$, monotonicity of $\mathbf{c}_{p^\sigma}$ implies that $\mathbf{c}_{p^\sigma}(y^\sigma_s, n^\sigma_s) \geqslant \varepsilon^*$. Thus by Lemma 2.4, between any two σ-action stages, the partial cost $\mathbf{c}_p(\langle A_{h_e}\rangle)[s]$ grows by at least ε^*, so eventually grows beyond 1, which is a contradiction. □

Now fix some $p \in \mathbb{Q} \cap (0, 1)$.

Lemma 2.7. Suppose that $p < r$. Then for all e, the requirement $R_{p,e}$ is met.

Proof. By Lemma 2.3(b), let $\sigma \in \delta^*$ such that $(p^\sigma, e^\sigma) = (p, e)$. Since $p < r$, $\sigma\hat{\ }\mathtt{fin} \in \delta^*$. Then Lemma 2.6 implies that $R_{p,e}$ is met. □

Lemma 2.8. Suppose that $p > r$. Then the requirement N_p is met.

Proof. Let σ be the longest node on the true path such that $r^\sigma > p$. So $p^\sigma \leqslant p$ and $\sigma\hat{\ }\infty \in \delta^*$. Let s^* be sufficiently late so that:

- σ is not initialised after stage s^*; and
- For every τ such that $\tau\hat{\ }\texttt{fin} \leqslant \sigma$, there are no τ-action stages after stage s^*;

the latter uses Lemma 2.6. Let $s_0 < s_1 < s_2 < \dots$ be the increasing enumeration of the $\sigma\hat{\ }\infty$-stages after stage s^*. We show that $\mathbf{c}_{p^\sigma}\langle A_{s_k}\rangle$ is finite, which suffices since $p^\sigma \leqslant p$.

Let $k \geqslant 1$; let x_k be the least such that $A_{s_k}(x_k) \neq A_{s_{k-1}}(x_k)$. Let τ_k be the node such that $x_k \in \omega^{[\tau_k]}$. So there is some τ_k-action stage $t_k \in [s_{k-1}, s_k)$ such that $x_k = y^{\tau_k}_{t_k}$. Since $t_k > s^*$, we know that $\tau_k\hat{\ }\texttt{fin}$ lies to the right of $\sigma\hat{\ }\infty$, or τ_k extends $\sigma\hat{\ }\infty$. In the first case (which includes the case $\tau_k = \sigma$), $\tau_k\hat{\ }\texttt{fin}$ is initialised at stage s_{k-1}, and so $x_k > s_{k-1} \geqslant k$, which implies that $\mathbf{c}_{p^\sigma}(x_k, k) = 0$; so stage k contributes no cost to the total cost $\mathbf{c}_{p^\sigma}\langle A_{s_k}\rangle$.

Suppose that τ_k extends $\sigma\hat{\ }\infty$. Then $t_k = s_{k-1}$, and more importantly, $r^{\tau_k} \leqslant r^{\sigma\hat{\ }\infty} = p^\sigma$. Thus

$$\mathbf{c}_{p^\sigma}(x_k, k) \leqslant \mathbf{c}_{r^\tau}(x_k, s_{k-1}) \leqslant \mathbf{c}_{p^\tau}(x_k, n^\tau_{s_{k-1}}) \cdot \delta^\tau.$$

It follows that

$$\mathbf{c}_{p^\sigma}\langle A_{s_k}\rangle = \sum_k \mathbf{c}_{p^\sigma}(x_k, k) \leqslant \sum_{\tau \geqslant \sigma\hat{\ }\infty} \delta^\tau \cdot \sum \{\mathbf{c}_{p^\tau}(y^\tau_s, n^\tau_s) \,:\, s \text{ a } \tau\text{-action stage}\} \leqslant \sum_\tau 2\delta^\tau \leqslant 2$$

(using Lemma 2.5), and so is finite as required. □

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