COMPUTING K-TRIVIAL SETS BY
INCOMPLETE RANDOM SETS

LAURENT BIENVENU, ADAM R. DAY, NOAM GREENBERG, ANTONÍN KUČERA,
JOSEPH S. MILLER, ANDRÉ NIES, AND DAN TURETSKY

ABSTRACT. Every K-trivial set is computable from an incomplete Martin-Löf random set, i.e., a Martin-Löf random set that does not compute the halting problem.

A major objective in algorithmic randomness is to understand how random sets and computably enumerable (c.e.) sets interact within the Turing degrees. At some level of randomness all interesting interactions cease. The lower and upper cones of noncomputable c.e. sets are definable null sets, and thus if a set is "sufficiently" random, it cannot compute, nor be computed by, a noncomputable c.e. set. However, the most studied notion of algorithmic randomness, Martin-Löf randomness, is not strong enough to support this argument, and in fact, significant interactions between Martin-Löf random sets and c.e. sets occur. The study of these interactions has lead to a number of surprising results that show a remarkably robust relationship between Martin-Löf random sets and the class of K-trivial sets. Interestingly, the significant interaction occurs "at the boundaries": the Martin-Löf random sets in question are close to being non-random (in that they fail fairly simple statistical tests), and K-trivial c.e. sets are close to being computable.

The following theorem resolves one of the main open questions in algorithmic randomness, and further strengthens the relationship between the Martin-Löf random sets and the K-trivial sets.

Theorem 1. *There is an incomplete Martin-Löf random set that computes every K-trivial set.*

This theorem is essentially a corollary of two recent results, both proved in 2012: the first by Bienvenu, Greenberg, Kučera, Nies and Turetsky [3]; and the second by Day and Miller [10]. In the remainder of this announcement, we will explain the background to the problem behind this theorem, and indicate the main ideas used in the proof.

In this announcement, by "random" we will henceforth mean Martin-Löf random. We will give the full definition shortly, but essentially, an element X of Cantor space is Martin-Löf random if it is not an element of a particular kind of effectively

presented, null G_δ class.[1] Any random set computes a diagonally noncomputable function. This implies, by Arslanov's completeness criterion [1], that the only c.e. sets that can compute random sets are the complete ones. Two questions of interest are thus:

(1) classifying those random sets that can compute incomputable c.e. sets, and
(2) classifying those c.e. sets that can be computed by random sets.

In 1986, Kučera [21] showed that every Δ_2^0 random set is Turing above a noncomputable c.e. set, demonstrating that these questions may have interesting answers.

The first question was settled by Hirschfeldt and Miller (see [30, Th. 5.3.16]). To explain their answer we need to provide some definitions. We denote Cantor space by 2^ω. An effectively presented G_δ set is the intersection $\bigcap_n \mathcal{U}_n$ of a nested, computable sequence of effectively open (Σ_1^0) subsets of 2^ω (i.e., it is a Π_2^0 class). A G_δ set E is *null* if $\lambda(E) = 0$, where λ denotes the fair-coin measure on Cantor space, which is mapped to Lebesgue measure under the standard near-isomorphism between Cantor space and the unit interval $[0, 1]$. Those elements of Cantor space that are not contained in any null, effectively presented G_δ sets are, in the parlance of the field, known as *weakly 2-random*. A set $X \in 2^\omega$ is *Martin-Löf random* if it is not a member of any effectively presented G_δ set $\bigcap_n \mathcal{U}_n$ with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$. This additional condition specifies that not only is $\bigcap_n \mathcal{U}_n$ null, but this fact is witnessed in a strong manner. Such a null set is known as a *test* for Martin-Löf randomness. Martin-Löf randomness can be defined by other means, for example using effective betting strategies or compressibility of initial segments.[2] Hirschfeldt and Miller showed that a Martin-Löf random set X computes a noncomputable c.e. set if and only if it is not weakly 2-random. This gives a pleasing characterization of those "special" random sets that compute noncomputable c.e. sets using the tools of effective measure theory (or very low level effective descriptive set theory).

For the second question, we are faced with a simple example: *all* c.e. sets are computable from some random set; indeed a single random set, Chaitin's "halting probability" Ω, computes all c.e. sets. In fact, the Kučera–Gács theorem [20, 17] states that every set is computable from a random set. However, this is only a partial answer to the question. There is a dichotomy between two kinds of random sets. On the one hand, those randoms that compute $\varnothing'$, the *complete* ones, pass all the relevant statistical tests (the effective null classes) not because they are in some way typical, but because their strong information content allows them mimic typical sets. On the other hand, *incomplete* random sets, those that do not compute $\varnothing'$, are "more random" in that they lack significant computational power. Stephan proved that an incomplete random set cannot even compute a complete extension of Peano arithmetic [34]. In this announcement, we characterize the c.e. sets covered by, which means Turing computable from, incomplete random sets. We will see that the only c.e. sets that can be computed by incomplete random sets are very weak, i.e., share many properties with the computable sets [19]. We begin by giving some background on the appropriate instance of computational weakness, K-triviality.

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[1]Elements of Cantor space are called *reals*, *sequences* or simply *sets* by computability theorists, the latter since they are identified with subsets of ω. Subsets of Cantor space are referred to as either *sets* or *classes*.

[2]For more on algorithmic randomness, the reader can consult the books [30, 12], which are the most up-to-date surveys of the field.

The K-trivial sets were introduced by Chaitin [8] and first studied by Solovay in an unpublished manuscript [33]. One motivation for their definition was Chaitin's characterization [7] of the computable sets by the compressibility of their initial segments. Namely, letting C denote plain Kolmogorov complexity, the sets A that satisfy $C(A \upharpoonright_n) \leqslant^+ C(n)$ are the computable ones.[3] The condition means that every initial segment of A contains no more information than its length, and so it is as compressible as it can be.

The use of prefix-free Kolmogorov complexity K, rather than its plain variant C, is motivated by Schnorr's theorem ([32], see [6]), which characterizes the random sets as those that have K-incompressible initial segments: a set A is random if and only if $K(A \upharpoonright_n) \geqslant^+ n$. The measure of an open set is determined by any prefix-free set of finite strings generating it, and so the study of prefix-free complexity, unlike the plain variety, is closely linked with measure-theoretic arguments. This explains why K is the most useful Kolmogorov complexity when considering the randomness content of sets. The K-trivial sets—those sets A whose initial segments are as K-compressible as possible, in that $K(A \upharpoonright_n) \leqslant^+ K(n)$—are the very opposite of random sets, and for that reason have sometimes been called "anti-random". Surprisingly, Solovay showed that in contrast with plain complexity, there are non-computable K-trivial sets.

Chaitin [8] showed that all K-trivial sets are Δ_2^0, that is, computable from the halting problem $\varnothing'$. In the 2000's, a series of results greatly improved our understanding of the K-trivial sets. These results showed that the K-trivial sets are all computationally very weak, that the class of K-trivial sets is robust, and that the ideal of K-trivial sets is generated by its c.e. elements.

The computational weakness of the K-trivial sets was first indicated by Downey, Hirschfeldt, Nies and Stephan [13], who showed that K-trivial sets are incomplete; that is, they do not compute $\varnothing'$. This result was then strengthened by Nies [29], who showed that in fact every K-trivial set is jump-traceable and superlow, roughly saying that in terms of the Turing jump operator, these sets are indistinguishable from the computable ones. Other ways to express their computational weakness in fact led [29] to equivalent definitions of the K-trivial sets. The K-trivial sets coincide with the sets that are *low for K*—sets that have no compression power as oracles, in that K^A, the prefix-free complexity relative to A, is equal, up to an additive constant, to the unrelativized K. Similarly, the K-trivial sets coincide with the sets that are *low for ML-randomness*—sets A that cannot detect any regular patterns in random sets, in that every random set is also random relative to A. These equivalences witness the robustness of the class. This robustness is further reflected in the structure of the Turing degrees of K-trivial sets: these degrees form an ideal, and restricted to the c.e. sets, this ideal has a Σ_3^0 index set. Barmpalias and Nies showed that this ideal has a low$_2$ c.e. upper bound. Kučera and Slaman [24] provided a low upper bound that, however, cannot be c.e. by a result in [28] (also see [30, 5.3.22]).

The relation between K-triviality and enumerability began with a construction, by Zambella [35], of a noncomputable c.e. K-trivial set. Another such construction was given by Downey, Hirschfeldt, Nies and Stephan [13], which came to be known as the "cost function" construction; this construction was inspired by an earlier

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[3]By $f(n) \leqslant^+ g(n)$ we abbreviate $f(n) \leqslant g(n) + O(1)$. We also write $f(n) \geqslant^+ g(n)$ with the same meaning.

construction of a noncomputable c.e. set that is low for ML-randomness due to
Kučera and Terwijn [25]. Cost functions are used to gauge the complexity of a
Δ_2^0 set by the amount of changes needed for a computable approximation; for
background see, for instance, [30, Section 5.3]. Finally, Nies [29] showed that K-
triviality is an inherently enumerable notion: every K-trivial set is computable
from a c.e. K-trivial set (so in the Turing degrees, the ideal of K-trivial degrees is
generated by its c.e. elements). He also showed that the class of K-trivial sets can
be characterized in terms of cost functions.

These results have had several applications. Following the example of Kučera
[21], in [13] it was shown how c.e. K-trivial sets provide an injury-free solution to
Post's problem. A more surprising application was given by Kučera and Slaman
[23], who used K-trivial sets in their proof that no Scott set of Turing degrees is
"hourglass-like". That is, for every noncomputable real in the Scott set there is a
Turing incomparable real.

We say that a set A is a *base for randomness* if A is computable from some
A-random set. This notion was first studied in [22]. A noncomputable base for
randomness is so computationally weak that the cone above it, while being null,
is not an effective A-null class. A first indication of the relationship between K-
triviality and incomplete random sets came from work of Hirschfeldt, Nies and
Stephan [19]. They showed that if A is a c.e. set that is computable from an
incomplete random set X, then X is in fact random relative to A, and so A is
a base for randomness. In turn, they showed that the sets that are a base for
randomness coincide with the K-trivial sets, establishing the following theorem.

Theorem 2 ([19]). *Every c.e. set that is computable from an incomplete random
set is K-trivial.*

In light of this result, in 2004, Stephan asked whether the c.e. sets that are com-
putable from incomplete random sets are precisely the K-trivial sets. This problem,
known as the random covering problem, became one of the major open questions
in the field of algorithmic randomness. It is problem 4.6 in Miller and Nies's survey
[27] of open questions in the field. In light of the inherent enumerability of K-trivial
sets, this is equivalent to asking whether every K-trivial set is computable from an
incomplete random set. As already mentioned, Theorem 1 answers the question in
the affirmative. This gives yet another characterization of K-triviality, one which
uses only very basic ingredients from computability theory—namely computable
enumerability, Turing reducibility, and the halting problem—together with ML-
randomness (but with no reference to relativization).

In retrospect, the first step toward solving the covering problem was made by
Franklin and Ng who gave a characterization of the incomplete random sets [15].
This is analogous to the Hirschfeldt–Miller result in that it gives a measure-theoretic
characterization of a class of random sets defined by their interaction within the
Turing degrees. They formulated a notion of randomness—*difference randomness*—
and showed that it is equivalent to being random and Turing incomplete. In more
detail, they showed the equivalence, for a set Z, of the following properties:

(1) Z is random and incomplete.
(2) Z avoids all null sets of the form $\mathcal{P} \cap \bigcap_n \mathcal{U}_n$, where the $\mathcal{U}_n$ are uniformly
    effectively open, $\mathcal{P}$ is effectively closed, and $\lambda(\mathcal{P} \cap \mathcal{U}_n) \leqslant 2^{-n}$.

Franklin and Ng also showed that difference randomness could be characterized using a concept similar to a test for Martin-Löf randomness. In his investigations of differentiability of constructive functions on the reals, Demuth [11] introduced notions of randomness stronger than Martin-Löf's. Like ML-randomness, his tests are null sets that are the intersection of a sequence $\langle \mathcal{U}_n \rangle$ of effectively open sets with $\lambda(\mathcal{U}_n) \leqslant 2^{-n}$. However, when defining the sets $\mathcal{U}_n$ we are allowed to change our mind sometimes about what $\mathcal{U}_n$ is. In Martin-Löf tests, the function taking n to an index for $\mathcal{U}_n$ is computable; Demuth allowed effectively approximable functions, with a computable bound on the number of mind-changes;[4] this notion of randomness is now known as *weak Demuth* randomness (see [26] for background on Demuth's work in randomness). Franklin and Ng showed that a particular class of weak Demuth tests also captured difference randomness; in their tests, the different "versions" for each component $\mathcal{U}_n$ have to be disjoint.

The second step was made in 2011 by Bienvenu, Hölzl, Miller and Nies [4], who gave an analytic characterization of incomplete randomness. Recall that the Lebesgue density theorem says that if B is a measurable set, then for almost all $x \in B$, the limit of the conditional probability (or measure)

$$\lambda(B|I) = \frac{\lambda(B \cap I)}{\lambda(I)}$$

for intervals I that contain x of shrinking diameter, is 1. When working in Cantor space, rather that using arbitrary intervals, it is more natural to use dyadic intervals. The analog of Lebesgue's theorem in this context says that for any measurable set $B \subseteq 2^\omega$, for almost all $X \in B$, the *lower dyadic density of B at X*,

$$\rho(B|X) = \liminf_{n \to \infty} \lambda(B|[X \restriction_n]),$$

is 1; here $[\sigma]$ denotes the basic clopen subset of 2^ω determined by the finite binary string σ.

Computability theorists often try to find effective content in results of classical mathematics. In analysis, the effective versions of almost-everywhere theorems often translate to characterizations of notions of randomness in analytic terms. Bienvenu, Hölzl, Miller and Nies [4] applied this to a restricted form of Lebesgue's theorem and showed that the following are equivalent for a random set $X \in 2^\omega$:[5]

    (1) X is difference random;
    (2) $\rho(\mathcal{P}|X) > 0$ for all effectively closed sets $\mathcal{P}$ containing X.

If X has property (2), we call it a *positive density point*. Together with Franklin and Ng's work, we see that a random set X is incomplete if and only if it is a positive density point. The first indication that the analytic notion of density is relevant to understanding the interaction of random and K-trivial sets was given by Day and Miller [9]. They used density and the results from [4] to solve a problem related to the covering problem, known as the ML-cupping problem; in particular,

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[4]Formally, these are tests of the form $[W_{f(n)}]$ where f is the limit of a uniformly computable sequence of functions f_s such that for some computable function g, for all n, $|\{s \colon f_{s+1}(n) \neq f_s(n)\}| \leqslant g(n)$.

[5]They actually showed the result for Lebesgue density in the context of the unit interval, and then derived the weaker dyadic variant (for detail see the journal version [5, Remark 3.4]); we restrict our attention to dyadic density.

they showed that a set A is K-trivial if and only if there is no incomplete random set Z that joins A above $\varnothing'$.

The work in [4] left open the problem of characterizing those random sets X for which the full effective version of Lebesgue's density theorem holds. We say that $X \in 2^\omega$ is a *density-one point* if $\rho(\mathcal{P}|X) = 1$ for all effectively closed sets $\mathcal{P}$ containing X. The question that remained was whether every random positive density point is a density-one point. Put differently, if X is random and not a density-one point, must it compute $\varnothing'$? It was known by July 2011 (see Bienvenu, Hölzl, Miller and Nies [5]) that any such set is LR-*hard*: every X-random set is actually $\varnothing'$-random. While this proves that X has much of the computational strength of $\varnothing'$, it was also known ([30, 6.3.10], also Kučera (unpublished)) that some incomplete random sets have this highness property.

During and after a research-in-pairs stay at the mathematical research institute in Oberwolfach in February 2012, Bienvenu, Greenberg, Kučera, Nies and Turetsky found the analogue of the Hirschfeldt–Miller and Franklin–Ng characterizations for computing K-trivial sets. They defined a notion of randomness, called *Oberwolfach randomness*. They were motivated by work by Figueira, Hirschfeldt, Miller, Ng and Nies [14], who investigated the randomness strength of a Δ_2^0 random set Z by counting the number of changes required in any computable approximation of Z. This was linked with Demuth's idea, mentioned earlier, of accepting changes in components of tests. Oberwolfach randomness is a weak form of weak Demuth randomness, in which the changes of the components have to be coherent between the levels (the changes of $\mathcal{U}_{n+1}$ are limited by the changes of $\mathcal{U}_n$).[6] In fact, examining the argument given by Franklin and Ng shows that a "version-disjoint" variant of Oberwolfach randomness suffices in order to capture difference randomness. Thus, Oberwolfach randomness lies between weak Demuth and difference randomness. And indeed, this notion of randomness characterized the candidates for the solution of the covering problem.

Theorem 3 (Bienvenu, Greenberg, Kučera, Nies and Turetsky [3])**.**

(1) *Suppose a set X is random. Then X is not Oberwolfach random if and only if X computes all K-trivial sets.*

(2) *There is a K-trivial set that is "smart" in that it is not computable from any Oberwolfach random set.*

The "smart" K-trivial showed that if the covering problem has a positive solution, then it has a strong positive solution in that some incomplete random set would compute *all* K-trivial sets.

For the solution of the covering problem, one needs the implication from left to right in (1). The authors show that for any random set X that is not Oberwolfach random, there is a cost function c, such that X computes any c.e. set that obeys c. Further, all K-trivial sets obey c.[7]

The authors of [3] used the technique from [5] to show that every random that is not Oberwolfach random is LR-hard, and therefore high. Thus, the construction of the "smart" K-trivial set showed that no low random set can compute all K-trivial

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[6]Formally, this is a weak Demuth test $[W_{f(n)}]$ such that for all n, if the interval (s,t) is maximal such that there are no changes to $f_r(n+1)$ for any r in this interval, then either $f_{s+1}(n) \neq f_s(n)$ or $f_{t+1}(n) \neq f_t(n)$.

[7]The cost functions used here are known as additive (see [3, 31]).

sets (in contrast with the existence of a low PA degree above all K-trivial sets). The question whether such a random set exists was a strong variant of the covering problem, which was also posed by Stephan in 2004. Using work from [14], they also concluded that the smart K-trivial set is not computable from both halves of a random set, negatively solving another strong variant of the covering problem (Problem 4.7 in [27]).

Further, the authors of [3] made a connection between Oberwolfach randomness and Lebesgue's density theorem, by showing that if X is Oberwolfach random then it is a density-one point. This was a corollary of showing that if X is Oberwolfach random, then every interval-c.e.[8] function is differentiable at X (identifying X with the real that has binary expansion $0.X$). As a by-product, they obtained:

Theorem 4 ([3])**.** *If X is a random set that is not a density-one point, then X computes all K-trivial sets.*

An alternative proof of Theorem 4 is given in [5], using the same technique as the proof that such an X is LR-hard. However, the alternative proof requires the fact that K-trivial sets are low for ML-randomness, hence it is not useful in the proof of Theorem 7 below.

Theorem 4 set the stage for the last ingredient in the solution of the covering problem, which was provided in August 2012 by Day and Miller [10]. Devising a notion of forcing using a collection of effectively closed sets especially defined to control density, they showed:

Theorem 5 ([10])**.** *There is a random set X (in fact, a Δ_2^0 one) that is a positive density point but not a density-one point.*

The Franklin–Ng, Bienvenu–Hölzl–Miller–Nies, Bienvenu–Greenberg–Kučera–Nies–Turetsky and Day–Miller results now all combine to settle the covering problem in the affirmative:

Corollary 6. *There is an incomplete, Δ_2^0 random set that computes all K-trivial sets.*

Further, Day and Miller incorporated non-K-trivial upper-cone avoidance with their forcing and constructed an exact pair of random sets for the ideal of K-trivial degrees.

Apart from the inherent interest in the covering problem, and the unexpected path to its solution using analytic concepts, these results give alternative, modular proofs of some of the results concerning K-trivial sets. Hitherto, the fact that K-trivial sets are low for K (or low for ML-randomness), and the fact that the K-trivial sets are downward closed under Turing reducibility, were proved using the *decanter* argument and its stronger version, the *golden run* technique. Researchers have found these highly combinatorial and complex techniques somewhat daunting. We can now provide alternative arguments. We note that (2) below easily implies (1).

Theorem 7 (Nies [29])**.**

(1) *Let A be K-trivial. Then every set $B \leqslant_{\mathrm{T}} A$ is K-trivial.*

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[8]A non-decreasing lower semicontinuous function $f \colon [0,1] \to \mathbb{R}$ is *interval-c.e.* if $f(0) = 0$ and $f(y) - f(x)$ is a left-c.e. real, uniformly in rational numbers x, y. By work of Freer, Kjos-Hansen, Nies and Stephan [16], the continuous interval c.e. functions coincide with the variation functions of computable real-valued functions.

(2) *Let A be K-trivial. Then A is low for K (hence low for ML-randomness).*

Proof. (1) A direct construction of Bienvenu's (see a forthcoming journal paper with Downey, Merkle and Nies related to [2]) shows that A is computable from some c.e., K-trivial set C. In [3] it is proved directly that every c.e., K-trivial set is computable from every random set that is not Oberwolfach random. Then, by the argument above, C is computable from some incomplete random set Z. Now the Hirschfeldt–Nies–Stephan argument first shows that Z is C-random, and so certainly A-random and B-random. Thus, A and a fortiori B, is a base for randomness. The "hungry sets" argument from [19] now shows that B is K-trivial. (2) The set A is low for K by a slightly more elaborate "hungry sets" argument due to [30, Section 5.1]. $\square$
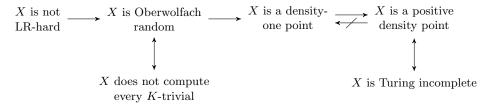
Diamond classes were investigated in [30], [18] and elsewhere. These are collections of c.e. sets of the form

$$\mathcal{C}^\diamond = \{A \text{ c.e.} : A \leqslant_{\mathrm{T}} X \text{ for all random } X \in \mathcal{C}\},$$

where $\mathcal{C}$ is any collection of sets; they are naturally ideals in the c.e. degrees. If $\mathcal{C}$ is not null, then $\mathcal{C}^\diamond$ consists of the computable sets; the Hirschfeldt–Miller argument shows that if $\mathcal{C}$ is a null Σ_3^0 class then $\mathcal{C}^\diamond$ contains a noncomputable set. Greenberg and Nies [18] showed, for example, that the strongly jump-traceable c.e. sets form a diamond class. Results in [3], together with the smart K-trivial set, give a diamond class ($\mathsf{JTH}^\diamond$) that lies strictly between the K-trivial and the strongly jump-traceable degrees. The covering result answers a question by Nies, by letting $\mathcal{C}$ be the class of sets that are not Oberwolfach random:

Corollary 8. *The K-trivial c.e. sets form a diamond class.*

Let $X \in 2^\omega$ be Martin-Löf random. The following diagram summarizes the properties we have discussed:

As mentioned above, there is a Turing incomplete random that is LR-hard. By [10], there is a ML-random positive density point that is not a density-one point. Also, as mentioned already, the Turing incomplete ML-randoms coincide with the difference randoms [15]. The rightmost vertical equivalence is due to [4] as discussed above. The other nontrivial implications were obtained in [3].

Question 9. Determine which further implications hold for ML-random sets.

In particular, we ask the following.

Question 10. Is there an LR-hard Oberwolfach random set? Equivalently, does the collection of K-trivial sets strictly contain the diamond class $\mathsf{LRH}^\diamond$?

Question 11. What is the effective measure-theoretic characterization of the random sets for which Lebesgue's density theorem hold? Is it Oberwolfach randomness, or a weaker notion? How does it relate to LR-hardness?

References

[1] Marat M. Arslanov. Some generalizations of a fixed-point theorem. *Izv. Vyssh. Uchebn. Zaved. Mat.*, (5):9–16, 1981.

[2] Laurent Bienvenu and Rodney G. Downey. Kolmogorov complexity and Solovay functions. In *Symposium on Theoretical Aspects of Computer Science (STACS 2009)*, volume 09001 of *Dagstuhl Seminar Proceedings*, pages 147–158, http://drops.dagstuhl.de/opus/volltexte/2009/1810, 2009.

[3] Laurent Bienvenu, Noam Greenberg, Antonín Kučera, André Nies, and Daniel Turetsky. *K*-triviality, Oberwolfach randomness, and differentiability. Mathematisches Forschungsinstitut Oberwolfach, Preprint Series, 2012.

[4] Laurent Bienvenu, Rupert Hoelzl, Joseph S. Miller, and André Nies. The Denjoy alternative for computable functions. In *STACS 2012*, pages 543–554, 2012.

[5] Laurent Bienvenu, Rupert Hoelzl, Joseph S. Miller, and André Nies. Demuth, Denjoy, and Density. Submitted, 2013.

[6] Gregory J. Chaitin. A theory of program size formally identical to information theory. *J. Assoc. Comput. Mach.*, 22:329–340, 1975.

[7] Gregory J. Chaitin. Information-theoretic characterizations of recursive infinite strings. *Theoret. Comput. Sci.*, 2(1):45–48, 1976.

[8] Gregory J. Chaitin. Nonrecursive infinite strings with simple initial segments. *IBM Journal of Research and Development*, 21:350–359, 1977.

[9] Adam R. Day and Joseph S. Miller. Cupping with random sets. To appear in the *Proc. Amer. Math. Soc.*

[10] Adam R. Day and Joseph S. Miller. Density, forcing, and the covering problem. In preparation.

[11] Osvald Demuth. Some classes of arithmetical real numbers. *Comment. Math. Univ. Carolin.*, 23(3):453–465, 1982.

[12] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic randomness and complexity*. Theory and Applications of Computability. Springer, New York, 2010.

[13] Rodney G. Downey, Denis R. Hirschfeldt, André Nies, and Frank Stephan. Trivial reals. In *Proceedings of the 7th and 8th Asian Logic Conferences*, pages 103–131, Singapore, 2003. Singapore Univ. Press.

[14] Santiago Figueira, Denis Hirschfeldt, Joseph S. Miller, Keng Meng Ng, and André Nies. Counting the changes of random Δ_2^0 sets. *CiE 2010*, 2010. June 2010 available electronically at http://link.springer.com/chapter/10.10072F978-3-642-13962-8-18page-1.

[15] Johanna N. Y. Franklin and Keng Meng Ng. Difference randomness. *Proc. Amer. Math. Soc.*, 139(1):345–360, 2011.

[16] Cameron Freer, Bjørn Kjos-Hanssen, André Nies, and Frank Stephan. Effective aspects of Lipschitz functions. Submitted.

[17] Peter Gács. Every sequence is reducible to a random one. *Inform. and Control*, 70:186–192, 1986.

[18] Noam Greenberg and André Nies. Benign cost functions and lowness properties. *J. Symbolic Logic*, 76(1):289–312, 2011.

[19] Denis R. Hirschfeldt, André Nies, and Frank Stephan. Using random sets as oracles. *J. Lond. Math. Soc. (2)*, 75(3):610–622, 2007.

[20] Antonín Kučera. Measure, Π_1^0-classes and complete extensions of PA. In *Recursion theory week (Oberwolfach, 1984)*, volume 1141 of *Lecture Notes in Math.*, pages 245–259. Springer, Berlin, 1985.

[21] Antonín Kučera. An alternative, priority-free, solution to Post's problem. In *Mathematical foundations of computer science, 1986 (Bratislava, 1986)*, volume 233 of *Lecture Notes in Comput. Sci.*, pages 493–500. Springer, Berlin, 1986.

[22] Antonín Kučera. On relative randomness. *Ann. Pure Appl. Logic*, 63(1):61–67, 1993. 9th International Congress of Logic, Methodology and Philosophy of Science (Uppsala, 1991).

[23] Antonín Kučera and Theodore A. Slaman. Turing incomparability in Scott sets. *Proc. Amer. Math. Soc.*, 135(11):3723–3731, 2007.

[24] Antonín Kučera and Theodore A. Slaman. Low upper bounds of ideals. *Journal of Symbolic Logic*, 74:517–534, 2009.

[25] Antonín Kučera and Sebastiaan A. Terwijn. Lowness for the class of random sets. *Journal of Symbolic Logic*, 64:1396–1402, 1999.

[26] Antonín Kučera and André Nies. Demuth's path to randomness. In *Proceedings of the 2012 international conference on Theoretical Computer Science: computation, physics and beyond*, WTCS'12, pages 159–173, Berlin, Heidelberg, 2012. Springer-Verlag.

[27] Joseph S. Miller and André Nies. Randomness and computability: open questions. *Bull. Symbolic Logic*, 12(3):390–410, 2006.

[28] A. Nies. Reals which compute little. In *Logic Colloquium '02*, Lecture Notes in Logic, pages 260–274, 2002.

[29] André Nies. Lowness properties and randomness. *Adv. Math.*, 197(1):274–305, 2005.

[30] André Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009.

[31] André Nies. Interactions of computability and randomness. In *Proceedings of the International Congress of Mathematicians*, pages 30–57. World Scientific, 2010.

[32] Claus-Peter Schnorr. Process complexity and effective random tests. *J. Comput. System Sci.*, 7:376–388, 1973. Fourth Annual ACM Symposium on the Theory of Computing (Denver, Colo., 1972).

[33] Robert M. Solovay. Draft of paper (or series of papers) related to Chaitin's work. IBM Thomas J. Watson Research Center, Yorktown Heights, NY, 215 pages, 1975.

[34] Frank Stephan. Martin-Löf random and PA-complete sets. In *Logic Colloquium '02*, volume 27 of *Lect. Notes Log.*, pages 342–348. Assoc. Symbol. Logic, La Jolla, CA, 2006.

[35] Domenico Zambella. On sequences with simple initial segments. ILLC technical report ML 1990-05, Univ. Amsterdam, 1990.

LIAFA, CNRS & Université Paris 7, Case 7014, 75205 Paris Cedex 13, France
E-mail address: `laurent.bienvenu@liafa.univ-paris-diderot.fr`

Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840, USA
E-mail address: `adam.day@math.berkeley.edu`

School of Mathematics, Statistics and Operations Research, Victoria University of Wellington, Wellington, New Zealand
E-mail address: `greenberg@msor.vuw.az.nz`

Charles University in Prague, Faculty of Mathematics and Physics, Prague, Czech Republic
E-mail address: `kucera@mbox.ms.mff.cuni.cz`

Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA
E-mail address: `jmiller@math.wisc.edu`

Department of Computer Science, Private Bag 92019, Auckland, New Zealand
E-mail address: `andre@cs.auckland.ac.nz`

Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Vienna, Austria
E-mail address: `turetsd4@univie.ac.at`