**Final Homework.** Surface integrals and the divergence theorem.

**I.** Find the area of the surface $\mathcal{S} = \{(x, y, z) \colon z = \sqrt{x^2 + y^2}, 0 \le z \le 1\}$.

**II.** Let $\mathcal{S} = \{(x, y, z) \colon 0 \le x \le 1, 0 \le y \le 1, z = xy\}$. Find

$$\iint_{\mathcal{S}} z \, dA.$$

**III.** Let $\vec{\boldsymbol{F}} = y^3 \vec{\boldsymbol{j}} + 3x^2(z+1)\vec{\boldsymbol{k}}$.

(a) If $\mathcal{S}_1 = \{(x, y, z) \colon 0 \le x \le 2, 0 \le y \le 1, z = 0\}$ and $\vec{\boldsymbol{N}}$ is the upward pointing unit normal, find the flux $\iint_{\mathcal{S}_1} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}} \, dA$.

(b) If $\mathcal{S}_2 = \{(x, y, z) \colon x^2 + y^2 \le 1, z = 0\}$ and $\vec{\boldsymbol{N}}$ is the downward pointing unit normal, find the flux $\iint_{\mathcal{S}_2} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}} \, dA$.

(c) If $\mathcal{S}_3$ is the boundary of the half ball $\mathcal{R} = \{(x, y, z) \colon 0 \le z \le \sqrt{1 - x^2 - y^2}\}$ and $\vec{\boldsymbol{N}}$ is the outward unit normal, use the divergence theorem to find the flux $\iint_{\mathcal{S}_3} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}} \, dA$.

(d) Let $\mathcal{S}_4 = \{(x, y, z) \colon z = \sqrt{1 - x^2 - y^2}\}$ and let $\vec{\boldsymbol{N}}$ be the upward pointing unit normal to $\mathcal{S}_4$. Using parts (b) and (c), find $\iint_{\mathcal{S}_4} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}} \, dA$.

**IV.** Let $\mathcal{S}$ be the boundary of the region $\mathcal{R} = \{(x, y, z) \colon -1 \le x \le 1, 0 \le y \le 3, 0 \le z \le 1\}$. Let $\vec{\boldsymbol{N}}$ be the outward unit normal to $\mathcal{S}$. Let $\vec{\boldsymbol{F}} = xyz\vec{\boldsymbol{\imath}} + xyz\vec{\boldsymbol{\jmath}} + yz^2\vec{\boldsymbol{k}}$. Use the divergence theorem to find the (outward) flux of $\vec{F}$ across $\mathcal{S}$, i.e.,

$$\iint_{\mathcal{S}} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}} \, dA.$$

**V.** Consider the cylindrical region $\mathcal{R} = \{(x, y, z) \colon x^2 + y^2 \le 1, 0 \le z \le 3\}$. Let $\mathcal{S}$ be the boundary of $\mathcal{R}$ and let $\vec{\boldsymbol{N}}$ be the outward unit normal to $\mathcal{S}$. Let $\vec{\boldsymbol{F}} = x\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}} + z^2\vec{\boldsymbol{k}}$. Find

$$\iint_{\mathcal{S}} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}} \, dA$$

in two ways:

(a) By directly calculating the flux across $\mathcal{S}$ (hint: beak it up into three surface patches),

(b) By using the divergence theorem.

## Answers.

**I.** $\sqrt{2}\,\pi$

**II.** $(3^{5/2} - 2 \cdot 2^{5/2} + 1)/15$. This is the result of the integral $\int_0^1 \int_0^1 uv\sqrt{v^2 + u^2 + 1}\, du\, dv$.

**III.**

(a) 8

(b) $-\frac{3}{4}\pi$ (note that we use the downward pointing unit normal)

(c) $\frac{4}{5}\pi$

(d) The flux across the boundary of the half ball is the sum of the flux across the top portion and the bottom portion. We found the flux across the bottom (with outward pointing normal!) in (b) and the total flux in (c), so the flux across the top portion is

$$\text{Total} - \text{Bottom} = \frac{4}{5}\pi - \left(-\frac{3}{4}\pi\right) = \frac{31}{20}\pi.$$

**IV.** 27/2

**V.** $15\pi$. When done directly, this breaks down into 0 for the bottom, $9\pi$ for the top, and $6\pi$ for the middle.