1. Consider the points $A = (1, 2, 3)$, $B = (1, -2, 2)$, and $C = (2, 1, 4)$.

   (a) [6 points] Find the area of the triangle formed by A, B, and C.

Solution: One way to solve this problem is to use the fact that $\|\overrightarrow{AB} \times \overrightarrow{AC}\|$ is the area of the parallelogram given by $\overrightarrow{AB}$ and $\overrightarrow{AC}$, so it is twice the area of the triangle. Note that

$$\overrightarrow{AB} = \begin{pmatrix} 1-1 \\ -2-2 \\ 2-3 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} 2-1 \\ 1-2 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{vmatrix} \vec{\boldsymbol{i}} & 0 & 1 \\ \vec{\boldsymbol{j}} & -4 & -1 \\ \vec{\boldsymbol{k}} & -1 & 1 \end{vmatrix} = -5\vec{\boldsymbol{i}} - 1\vec{\boldsymbol{j}} + 4\vec{\boldsymbol{k}} = \begin{pmatrix} -5 \\ -1 \\ 4 \end{pmatrix}.$$

So the area of the triangle is $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{(-5)^2 + (-1)^2 + 4^2} = \boxed{\sqrt{42}/2}$.

   (b) [6 points] Find t such that $D = (3, -2, t)$ is a point on the plane formed by A, B, and C.

Solution: We start by finding the plane containing A, B, and C. The vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is normal to this plane. So the equation will look like

$$-5x - y + 4z = k,$$

for some constant k. To find k, we can plug in one of the points, say A:

$$-5 \cdot 1 - 1 \cdot 2 + 4 \cdot 3 = 5 = k,$$

so the equation for the plane is $-5x - y + 4z = 5$. Now we plug D into this equation and solve for t:

$$\begin{aligned} -5 \cdot 3 - 1 \cdot (-2) + 4 \cdot t &= 5 \\ -13 + 4t &= 5 \\ 4t &= 18 \\ t &= \boxed{9/2}. \end{aligned}$$

2. (a) [6 points] Consider the quadratic form $f(x, y) = x^2 + Cxy + 2y^2$, where C is a constant. For which values of C is $f(x, y)$ indefinite?

Solution: Completing the square,

$$\begin{aligned} f(x,y) = x^2 + Cxy + 2y^2 &= (x + Cy/2)^2 - (Cy/2)^2 + 2y^2 \\ &= (x + Cy/2)^2 + (2 - C^2/4)y^2. \end{aligned}$$

So we see that $f(x, y)$ is indefinite when $2 - C^2/4 < 0$, i.e., when $C^2 > 8$. This happens when $\boxed{C < -2\sqrt{2} \text{ or } C > 2\sqrt{2}}$.

(b) [10 points] Consider the function $f(x, y) = x^3 + 6xy + 3y^2 - 9x$. Find all of the critical points of f. For each critical point, classify it as a local maximum, local minimum, or saddle point of f.

Solution: The critical points occur when the gradient is 0. Taking the gradient of f, we have

$$\vec{\nabla} f = \begin{pmatrix} 3x^2 + 6y - 9 \\ 6x + 6y \end{pmatrix},$$

so we have to solve the equations

$$\begin{aligned} 3x^2 + 6y - 9 = 0, \text{ and} \\ 6x + 6y = 0. \end{aligned}$$

From the second, we see that $y = -x$, so plugging into the first, we get $3x^2 - 6x - 9 = 0$. Therefore, $x = -1, 3$. So the critical points are $(-1, 1)$ and $(3, -3)$.

To classify the critical point (a, b) using the second derivative test, we have to analyze the quadratic form

$$Q(\Delta x, \Delta y) = \frac{1}{2}\left[f_{xx}(a,b)(\Delta x)^2 + 2f_{xy}(a,b)\Delta x\Delta y + f_{yy}(a,b)(\Delta y)^2\right].$$

Note that $f_{xx} = 6x$, $f_{xy} = 6$, and $f_{yy} = 6$. So the quadratic form becomes

$$\begin{aligned} Q(\Delta x, \Delta y) &= \frac{1}{2}\left[6a(\Delta x)^2 + 2 \cdot 6\Delta x\Delta y + 6(\Delta y)^2\right] \\ &= 3a(\Delta x)^2 + 6\Delta x\Delta y + 3(\Delta y)^2. \end{aligned}$$

For the critical point $(-1, 1)$, we have

$$\begin{aligned} Q(\Delta x, \Delta y) &= -3(\Delta x)^2 + 6\Delta x\Delta y + 3(\Delta y)^2 \\ &= -3\left[(\Delta x)^2 - 2\Delta x\Delta y - (\Delta y)^2\right] \\ &= -3\left[\left(\Delta x - \frac{2}{2}\Delta y\right)^2 - \left(-\frac{2}{2}\Delta y\right)^2 - (\Delta y)^2\right] \\ &= -3\left[(\Delta x - \Delta y)^2 - 2\,(\Delta y)^2\right], \end{aligned}$$

which is indefinite. Therefore, $\boxed{(-1,1) \text{ is a saddle point}}$.
For the critical point $(3,-3)$, we have

$$\begin{aligned} Q(\Delta x, \Delta y) &= 9(\Delta x)^2 + 6\Delta x \Delta y + 3(\Delta y)^2 \\ &= 3\left[(\Delta y)^2 + 2\Delta y \Delta x + 3(\Delta x)^2\right] \\ &= 3\left[\left(\Delta y + \frac{2}{2}\Delta x\right)^2 - \left(\frac{2}{2}\Delta x\right)^2 + 3(\Delta x)^2\right] \\ &= 3\left[(\Delta y + \Delta x)^2 + 2(\Delta x)^2\right], \end{aligned}$$

which is positive definite. (Note that we switched the usual roles of Δx and Δy in order to simplify the calculation slightly.) In any case, we see that $\boxed{(3,-3) \text{ is a local minimum}}$.

3. (a) [8 points] Consider the line segment joining the points $(1,2,3)$ and $(1,-2,2)$. Let $\vec{\boldsymbol{v}} = \begin{pmatrix} yz \\ x^2 \\ y+z \end{pmatrix}$. Find $\int_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{x}}$.

Solution: We parameterize the line segment using

$$\vec{\boldsymbol{x}}(t) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\left(\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 2-4t \\ 3-t \end{pmatrix}, \text{ for } 0 \le t \le 1.$$

Note that

$$\vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}(t)) \cdot \vec{\boldsymbol{x}}\,'(t) = \begin{pmatrix} (2-4t)(3-t) \\ 1^2 \\ (2-4t)+(3-t) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -4 \\ -1 \end{pmatrix} = -4 - (5-5t) = 5t - 9.$$

Therefore,

$$\int_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot d\vec{\boldsymbol{x}} = \int_0^1 \vec{\boldsymbol{v}}(\vec{\boldsymbol{x}}(t)) \cdot \vec{\boldsymbol{x}}\,'(t)\, dt = \int_0^1 5t - 9\, dt = 5/2 - 9 = \boxed{-\frac{13}{2}}.$$

(b) [5 points] Let $\vec{\boldsymbol{F}} = \begin{pmatrix} 2xe^y \\ x^2e^y + 1 \end{pmatrix}$. Find a function $z = f(x,y)$ such that $\vec{\boldsymbol{\nabla}} f = \vec{\boldsymbol{F}}$.

Solution: First, we integrate $f_x(x,y) = 2xe^y$ with respect to x, while treating y as a constant. This gives us

$$f(x,y) = x^2e^y + C(y),$$

where $C(y)$ is an unknown function of y. To determine C, we take the derivative with respect to y to get

$$f_y = x^2e^y + C'(y).$$

But we also know that $f_y = x^2e^y + 1$, so $C'(y) = 1$. Therefore, $C(y) = y + K$, where K is an arbitrary constant. Putting it all together,

$$\boxed{f(x, y) = x^2e^y + y + K},$$

where K is a constant. (We found all possible functions $f(x, y)$, even though we only needed to find one.)

(c) [5 points] Take $\vec{\boldsymbol{F}}$ as defined in part (b). Let $\mathcal{C}$ be a curve with initial point $(0, 5)$ and final point $(2, 3)$. Find $\displaystyle\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}}$.

Solution: By the fundamental theorem for line integrals,

$$\int_{\mathcal{C}} \vec{\boldsymbol{F}}\cdot d\vec{\boldsymbol{x}} = \int_{\mathcal{C}} (\vec{\boldsymbol{\nabla}} f)\cdot d\vec{\boldsymbol{x}} = f(B) - f(A),$$

where B is the final point and A is the initial point. So in our case we get

$$f(2, 3) - f(0, 5) = (2^2e^3 + 3) - (0^2e^5 + 5) = \boxed{4e^3 - 2}.$$

(Note that $\mathcal{C}$ was never specified; we are taking the line integral of a gradient, so the answer depends only on the initial and final points of $\mathcal{C}$.)

4. (a) [6 points] Let $\mathcal{R}$ be the region in the first quadrant of the plane bounded above by $y = 4 - x^2$. Let $\mathcal{C}$ be the boundary of $\mathcal{R}$ with counterclockwise orientation and let $\vec{\boldsymbol{v}} = \begin{pmatrix} xy \\ 2y \end{pmatrix}$. Use Green's theorem to find $\displaystyle\int_{\mathcal{C}} \vec{\boldsymbol{v}}\cdot d\vec{\boldsymbol{x}}$.

Solution: By Green's theorem,

$$\int_{\mathcal{C}} \vec{\boldsymbol{v}}\cdot d\vec{\boldsymbol{x}} = \iint_{\mathcal{R}} \frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(xy)\ dA = \iint_{\mathcal{R}} -x\ dA.$$

Note that $\mathcal{R} = \{(x, y)\colon 0 \le x \le 2, 0 \le y \le 4 - x^2\}$, so

$$\begin{aligned}\iint_{\mathcal{R}} -x\ dA &= \int_0^2 \int_0^{4-x^2} -x\ dy\, dx = \int_0^2 -xy\ \Big|_0^{4-x^2}\ dx \\ &= \int_0^2 -x(4 - x^2)\ dx = \int_0^2 x^3 - 4x\ dx = \left(x^4/4 - 2x^2\right)\Big|_0^2 \\ &= 2^4/4 - 2\cdot 2^2 = 4 - 8 = \boxed{-4}.\end{aligned}$$

(b) [6 points] Let $\mathcal{C}$ be as in part (a) and let $\vec{\boldsymbol{v}} = \begin{pmatrix} x^2 \\ xy \end{pmatrix}$. Find the outward flux of $\vec{\boldsymbol{v}}$ across $\mathcal{C}$, i.e., find $\int_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{N}}\, ds$, where $\vec{\boldsymbol{N}}$ is the outward pointing unit normal to $\mathcal{C}$.

Solution: By Green's theorem for flux integrals,

$$\int_{\mathcal{C}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{N}}\, ds = \iint_{\mathcal{R}} \operatorname{div}(\vec{\boldsymbol{v}})\, dA = \iint_{\mathcal{R}} \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy)\, dA = \iint_{\mathcal{R}} 3x\, dA.$$

Using our calculation from part (a),

$$\iint_{\mathcal{R}} 3x\, dA = -3 \iint_{\mathcal{R}} -x\, dA = -3(-4) = \boxed{12}.$$

(We could have done the calculation directly, but it would have been redundant.)

(c) [6 points] Let $\mathcal{R} = \{(x, y, z)\colon 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$. Let $\mathcal{S}$ be the boundary of $\mathcal{R}$ and let $\vec{\boldsymbol{N}}$ be the outward pointing unit normal to $\mathcal{S}$. Finally, let $\vec{\boldsymbol{v}} = \begin{pmatrix} 2x + \sin(z^2) \\ \cos(z^2) \\ yz \end{pmatrix}$. Find $\iint_{\mathcal{S}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{N}}\, dA$, i.e., the outward flux of $\vec{\boldsymbol{v}}$ across $\mathcal{S}$.

Solution: We will use the divergence theorem. First note that

$$\operatorname{div}(\vec{\boldsymbol{v}}) = \frac{\partial}{\partial x}\left(2x + \sin(z^2)\right) + \frac{\partial}{\partial y}\left(\cos(z^2)\right) + \frac{\partial}{\partial z}(yz) = 2 + y.$$

By the divergence theorem,

$$\begin{aligned}\iint_{\mathcal{S}} \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{N}}\, dA &= \iiint_{\mathcal{R}} \operatorname{div}(\vec{\boldsymbol{v}})\, dV = \iint_{\mathcal{R}} 2 + y\, dV \\ &= \int_0^1 \int_0^2 \int_0^3 2 + y\, dz\, dy\, dx = \int_0^1 \int_0^2 6 + 3y\, dy\, dx \\ &= \int_0^1 (6y + 3y/2) \Big|_0^2 dx = \int_0^1 18\, dx = \boxed{18}.\end{aligned}$$

5. [12 points] Let $\vec{\boldsymbol{F}} = \begin{pmatrix} y \\ x \\ z \end{pmatrix}$ and let $\mathcal{S}$ be the surface in three dimensional space given by $z = x^2$ for $-1 \le x \le 1$ and $0 \le y \le 2$. Find $\iint_{\mathcal{S}} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}}\, dA$, where $\vec{\boldsymbol{N}}$ is the unit normal to $\mathcal{S}$ that points upward (i.e., with positive z-coordinate).

Solution: We parameterize the surface patch using

$$\vec{\boldsymbol{x}}(u,v) = \begin{pmatrix} u \\ v \\ u^2 \end{pmatrix}, \text{ for } -1 \le u \le 1 \text{ and } 0 \le v \le 2.$$

Note that

$$\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v = \begin{pmatrix} 1 \\ 0 \\ 2u \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2u \\ 0 \\ 1 \end{pmatrix},$$

which has positive z-cooridinate, so it is pointing in the right direction (i.e., in the same direction as $\vec{\boldsymbol{N}}$). We also have

$$\vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}(u,v)) = \begin{pmatrix} v \\ u \\ u^2 \end{pmatrix}, \text{ so } \vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}(u,v)) \cdot (\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v) = -2uv + u^2.$$

Now we are ready to calculate the flux integral:

$$\begin{aligned}\iint_{\mathcal{S}} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{N}}\, dA &= \int_{-1}^{1} \int_0^2 \vec{\boldsymbol{F}}(\vec{\boldsymbol{x}}(u,v)) \cdot (\vec{\boldsymbol{x}}_u \times \vec{\boldsymbol{x}}_v)\, dv\, du = \int_{-1}^{1} \int_0^2 -2uv + u^2\, dv\, du \\ &= \int_{-1}^{1} (-uv^2 + u^2 v)\Big|_0^2\, du = \int_{-1}^{1} -4u + 2u^2\, du = (-2u^2 + 2u^3/3)\Big|_{-1}^{1} \\ &= (-2 + 2/3) - (-2 - 2/3) = \boxed{4/3}.\end{aligned}$$

6. [12 points] Find the surface area of the part of the paraboloid $z = x^2 + y^2$ for which $x^2 + y^2 \le 1$.

Solution: Note that if $x = r\cos\theta$ and $y = r\sin\theta$, then equation for the paraboloid becomes $z = x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$. So we can parameterize the relevant part of the paraboloid by using

$$\vec{\boldsymbol{x}}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{pmatrix}, \text{ for } 0 \le r \le 1 \text{ and } 0 \le \theta \le 2\pi.$$

Note that

$$\vec{\boldsymbol{x}}_r \times \vec{\boldsymbol{x}}_\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 2r \end{pmatrix} \times \begin{pmatrix} -r\sin\theta \\ r\cos\theta \\ 0 \end{pmatrix} = \begin{pmatrix} -2r^2\cos\theta \\ -2r^2\sin\theta \\ r\cos^2\theta + r\sin^2\theta \end{pmatrix} = r\begin{pmatrix} -2r\cos\theta \\ -2r\sin\theta \\ 1 \end{pmatrix},$$

so

$$\|\vec{\boldsymbol{x}}_r \times \vec{\boldsymbol{x}}_\theta\| = r\sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1^2} = r\sqrt{4r^2+1}.$$

Therefore, the surface area is

$$\iint_{\mathcal{S}} dA = \int_0^1 \int_0^{2\pi} r\sqrt{4r^2+1}\, d\theta\, dr = 2\pi \int_0^1 r\sqrt{4r^2+1}\, dr.$$

The integral on the right can be found using a substitution. Let $u = 4r^2 + 1$, so $du = 8r\, dr$. We have

$$2\pi \int_{x=0}^{x=1} r\sqrt{4r^2+1}\, dr = \frac{2\pi}{8} \int_{u=1}^{u=5} \sqrt{u}\, du = \frac{\pi}{4} \cdot \frac{u^{3/2}}{3/2}\Bigg|_1^5 = \boxed{\frac{\pi}{6}\left(5^{3/2}-1\right)}.$$

7. [12 points] Find the maximum and minimum values of $f(x, y) = x^2 + 2y^2 - x$ on the unit circle.

Solution: We use the method of Lagrange multipliers (but see below for an alternate solution that avoids this method). We are constrained to the unit circle (presumably, centered at the origin). If we let $g(x, y) = x^2 + y^2$, then this constraint can be expressed as $g(x, y) = 1$. We have

$$\vec{\nabla} f = \begin{pmatrix} 2x-1 \\ 4y \end{pmatrix} \text{ and } \vec{\nabla} g = \begin{pmatrix} 2x \\ 2y \end{pmatrix}.$$

Note that $\vec{\nabla} g = 0$ implies that $x, y = 0$, which is not consistent with the constraint. So we have to solve the equations

$$\begin{aligned} 2x - 1 &= 2x\lambda, \\ 4y &= 2y\lambda, \text{ and} \\ x^2 + y^2 &= 1. \end{aligned}$$

From the first, we get that either $y = 0$ or $\lambda = 2$.

Case 1: $y = 0$. In this case, the third equation gives us $x = \pm 1$.

Case 2: $\lambda = 2$. In this case, the first equation becomes $2x - 1 = 4x$, so $x = -1/2$. The third equation gives us $y = \pm\sqrt{1-x^2} = \pm\sqrt{1-(-1/2)^2} = \pm\sqrt{3}/2$.

We have found all of the possible points at which the extrema may occur. To finish, we check the values of $f(x, y)$ to find the maximum and minimum on the unit circle.

(x, y)	$f(x, y)$
$(-1, 0)$	2
$(1, 0)$	0
$(-1/2, -\sqrt{3}/2)$	2.25
$(-1/2, \sqrt{3}/2)$	2.25

Thus, $\boxed{\text{subject to the constraint } g(x,y)=1, \text{ the maximum value of } f(x,y) \text{ is } 2.25}$. It occurs at the points $(-1/2, \pm\sqrt{3}/2)$. The $\boxed{\text{minimum value is } 0}$. It occurs at the point $(1,0)$.

Alternate Solution: We don't actually have to use the method of Lagrange multipliers to solve this problem. On the unit circle we have $y^2 = 1 - x^2$, so

$$f(x,y) = x^2 + 2y^2 - x = x^2 + 2(1-x^2) - x = -x^2 - x + 2.$$

Since (x,y) is restricted to the unit circle, we know that $-1 \le x \le 1$. Therefore, all we have to do is find the extreme values of $h(x) = -x^2 - x + 2$ on the closed interval $[-1,1]$, which is a simple Calculus I problem. Note that $h'(x) = -2x - 1$ is only zero when $x = -1/2$, so that is the only critical point. So check the values of h at the critical point and the endpoints:

x	$h(x)$
-1	2
$-1/2$	2.25
1	0

The $\boxed{\text{minimum is } 0}$, which occurs when $x = 1$ and $y = \pm\sqrt{1-x^2} = \pm\sqrt{1-1^2} = 0$. The $\boxed{\text{maximum is } 2.25}$, which occurs when $x = -1/2$ and $y = \pm\sqrt{1-x^2} = \pm\sqrt{1-(-1/2)^2} = \pm\sqrt{3}/2$.