1. Suppose that $P(x,y) = Ax^2 + y^2$ and $Q(x,y) = Bx^2y + Cxy$, where A, B, and C are constants.

   (a) [7 points] For which values of the constants A, B, and C does there exist a function $f(x,y)$ such that $f_x = P$ and $f_y = Q$?

   Solution: There is a function $f(x,y)$ such that $f_x = P$ and $f_y = Q$ if and only if $P_y = Q_x$ (i.e., if and only if the hypothetical mixed partials are equal). So we need
   $$\frac{\partial P}{\partial y} = 2y = 2Bxy + Cy = \frac{\partial Q}{\partial x}.$$
   By matching up the coefficients on like terms, we see that this is only possible if $B = 0$ and $C = 2$. There is no restriction on A.

   (b) [7 points] Find such a function $f(x,y)$ for those values of A, B, and C for which it exists.

   Solution: From part (a), we have $P(x,y) = Ax^2 + y^2$ and $Q(x,y) = 2xy$, where A is arbitrary. At this point, it's actually pretty easy to guess a function $f(x,y)$ such that $f_x = P$ and $f_y = Q$, but for completeness, we will find the function systematically.

   First, if we integrate P with respect to x, treating y as constant, we find that
   $$f(x,y) = \frac{A}{3}x^3 + xy^2 + D(y),$$
   where $D(y)$ is an unknown function of y. Now differentiate with respect to y to get
   $$f_y = 2xy + D'(y).$$
   But if $f_y = Q = 2xy$, then $D'(y) = 0$, so $D(y)$ is a constant (with respect to both x and y). Let's call this constant K, so the final answer is
   $$\boxed{\frac{A}{3}x^3 + xy^2 + K, \text{ where } K \text{ is an arbitrary constant.}}$$

2. Consider the function $f(x,y) = 3xy - x^3 - y^3$.

   (a) [7 points] Find all of the critical points of f.

   Solution: The critical points occur when the gradient is 0, so taking the gradient of f, we have
   $$\vec{\nabla} f = \begin{pmatrix} 3y - 3x^2 \\ 3x - 3y^2 \end{pmatrix}.$$

So we have to solve the equations

$$\begin{aligned} 3y - 3x^2 &= 0, \text{ and} \\ 3x - 3y^2 &= 0. \end{aligned}$$

These reduce to $y = x^2$ and $x = y^2$. Plugging the first into the second, we get $y = y^4$. So either $y = 0$, or we can divide by y to get $1 = y^3$, hence $y = 1$. If $y = 0$, then $x = 0$ by the second equation. If $y = 1$, then $x = 1$.

So the critical points are $\boxed{(0,0) \text{ and } (1,1)}$.

(b) [7 points] For each critical point, classify it as a local maximum, local minimum, or saddle point of f.

Solution: To classify the critical point (a, b) using the second derivative test, we have to analyze the quadratic form

$$Q(\Delta x, \Delta y) = \frac{1}{2}\left[f_{xx}(a,b)(\Delta x)^2 + 2f_{xy}(a,b)\Delta x\Delta y + f_{yy}(a,b)(\Delta y)^2\right].$$

Note that $f_{xx} = -6x$, $f_{xy} = 3$, and $f_{yy} = -6y$. So the quadratic form becomes

$$\begin{aligned} Q(\Delta x, \Delta y) &= \frac{1}{2}\left[-6a(\Delta x)^2 + 2 \cdot 3\Delta x\Delta y - 6b(\Delta y)^2\right] \\ &= -3a(\Delta x)^2 + 3\Delta x\Delta y - 3b(\Delta y)^2. \end{aligned}$$

For the critical point $(0, 0)$, we have $Q(\Delta x, \Delta y) = 3\Delta x\Delta y$, which is clearly indefinite. Therefore, $\boxed{(0,0) \text{ is a saddle point}}$.

For the critical point $(1, 1)$, we have

$$\begin{aligned} Q(\Delta x, \Delta y) &= -3(\Delta x)^2 + 3\Delta x\Delta y - 3(\Delta y)^2 \\ &= -3\left[(\Delta x)^2 - \Delta x\Delta y + (\Delta y)^2\right] \\ &= -3\left[\left(\Delta x - \frac{1}{2}\Delta y\right)^2 - \left(-\frac{1}{2}\Delta y\right)^2 + (\Delta y)^2\right] \\ &= -3\left[\left(\Delta x - \frac{1}{2}\Delta y\right)^2 - \frac{1}{4}(\Delta y)^2 + (\Delta y)^2\right] \\ &= -3\left[\left(\Delta x - \frac{1}{2}\Delta y\right)^2 + \frac{3}{4}(\Delta y)^2\right], \end{aligned}$$

which is negative definite. Therefore, $\boxed{(1,1) \text{ is a local maximum}}$.

3. [15 points] Maximize $f(x, y) = xy$ subject to the constraint $8x^2 + y^2 = 1$.

Solution: We use the method of Lagrange multipliers. Let $g(x, y) = 8x^2 + y^2$, so the constraint is $g(x, y) = 1$. We have

$$\vec{\nabla} f = \begin{pmatrix} y \\ x \end{pmatrix} \text{ and } \vec{\nabla} g = \begin{pmatrix} 16x \\ 2y \end{pmatrix}.$$

Note that $\vec{\nabla} g = 0$ implies that $x, y = 0$, which is not consistent with the constraint. So we have to solve the equations

$$\vec{\nabla} f = \begin{pmatrix} y \\ x \end{pmatrix} = \lambda \begin{pmatrix} 16x \\ 2y \end{pmatrix} = \lambda \vec{\nabla} g,$$

along with the constrain equation, $g(x, y) = 1$. All together, we have to solve the system of equations

$$\begin{aligned} y &= 16x\lambda, \\ x &= 2y\lambda, \text{ and} \\ 8x^2 + y^2 &= 1. \end{aligned}$$

From the first two we get $y = 32y\lambda^2$. Note that if $y = 0$, then $x = 0$, which as we noted is inconsistent with the third equation. Therefore, we can divide by y to get $1 = 32\lambda^2$, or $\lambda = \pm\frac{1}{4\sqrt{2}}$.

Case 1: $\lambda = \frac{1}{4\sqrt{2}}$. Plugging λ into the first two equations makes them equivalent, so we are down to

$$\begin{aligned} y &= 2\sqrt{2}x, \text{ and} \\ 8x^2 + y^2 &= 1. \end{aligned}$$

Now this gives us $8x^2 + (2\sqrt{2}x)^2 = 16x^2 = 1$, so $x = \pm 1/4$. Plugging back into the first equation gives us the solutions $(1/4, 1/\sqrt{2})$ and $(-1/4, -1/\sqrt{2})$.

Case 2: $\lambda = -\frac{1}{4\sqrt{2}}$. Plugging λ into the first two equations makes them equivalent, so we are down to

$$\begin{aligned} y &= -2\sqrt{2}x, \text{ and} \\ 8x^2 + y^2 &= 1. \end{aligned}$$

Now this gives us $8x^2 + (-2\sqrt{2}x)^2 = 16x^2 = 1$, so $x = \pm 1/4$. Plugging back into the first equation gives us the solutions $(1/4, -1/\sqrt{2})$ and $(-1/4, 1/\sqrt{2})$.

We have found all of the possible points at which the maximum may occur. To finish, we check the values of $f(x, y)$ to figure out where it actually occurs.

(x, y)	$f(x, y)$
$\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{4\sqrt{2}}$
$\left(-\frac{1}{4}, -\frac{1}{\sqrt{2}}\right)$	$\frac{1}{4\sqrt{2}}$
$\left(\frac{1}{4}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{4\sqrt{2}}$
$\left(-\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$	$-\frac{1}{4\sqrt{2}}$

So the maximum possible value of $f(x,y)$ subject to the constraint $g(x,y) = 1$ is $\boxed{\frac{1}{4\sqrt{2}}}$. It occurs at the points $\boxed{\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{1}{4}, -\frac{1}{\sqrt{2}}\right)}$.

4. [15 points] Find the volume of the solid bounded by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

Solution: Note: It might be tempting to set this problem up in cylindrical coordinates, but as it turns out, that makes the calculation much worse.

From the first equation we get

$$-1 \leq x \leq 1, \text{ and } -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}.$$

The second equation gives us that

$$-\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}.$$

So the volume is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) dy\, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2}\; dy\, dx.$$

Note that

$$\begin{aligned} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2}\; dy &= \left(2\sqrt{1-x^2}\right) y \,\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \\ &= \left(2\sqrt{1-x^2}\right)\left(\sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right)\right) = 4\left(1-x^2\right). \end{aligned}$$

So the volume is

$$\begin{aligned} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2}\; dy\, dx &= \int_{-1}^{1} 4\left(1-x^2\right) dx = 4\left(x - x^3/3\right)\Big|_{-1}^{1} \\ &= 4(1-1/3) - 4((-1)-(-1)/3) = \boxed{16/3}. \end{aligned}$$

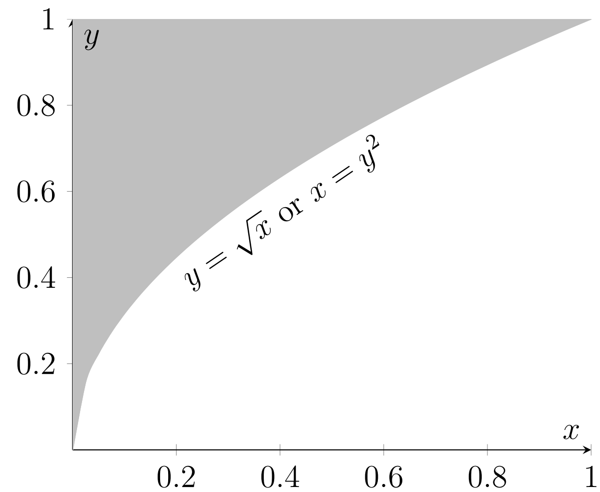
5. [14 points] Evaluate the following integral by *first changing the order of integration*:

$$\int_0^1 \int_{\sqrt{x}}^1 x e^{y^5} dy\, dx.$$

Solution: Note that

$$\int_0^1 \int_{\sqrt{x}}^1 x e^{y^5}\, dy\, dx = \iint_{\mathcal{D}} x e^{y^5}\, dA,$$

where $\mathcal{D}$ is the region pictured below.



Note that $\mathcal{D} = \{(x,y)\colon 0 \le y \le 1, 0 \le x \le y^2\}$, so

$$\iint_{\mathcal{D}} x e^{y^5}\, dA = \int_0^1 \int_0^{y^2} x e^{y^5}\, dx\, dy.$$

Calculating the inner integral, we get

$$\int_0^{y^2} x e^{y^5}\, dx = \frac{1}{2} x^2 e^{y^5} \Big|_0^{y^2} = \frac{1}{2} \left(y^2\right)^2 e^{y^5} = \frac{1}{2} y^4 e^{y^5}.$$

The outer integral is

$$\int_0^1 \frac{1}{2} y^4 e^{y^5}\, dy = \frac{1}{10} e^{y^5} \Big|_0^1 = \boxed{\frac{1}{10}(e-1)}.$$

6. [14 points] An object occupies the region inside the sphere of radius 3 and its density is given by $\mu(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$. Find its mass.

Solution: This problem is most easily solved using spherical coordinates. Let $\mathcal{S}$ be the sphere of radius 3 (centered at the origin).

$$\text{Mass} = \iiint_{\mathcal{S}} \mu \, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^3 \mu \, \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_0^{2\pi} \int_0^{\pi} \int_0^3 \rho \sin\varphi \, d\rho \, d\varphi \, d\theta,$$

using the fact that $\rho = \sqrt{x^2+y^2+z^2}$, so $\mu(x,y,z) = 1/\rho$. Calculating the inner integral, we get

$$\int_0^3 \rho \sin\varphi \, d\rho = \frac{1}{2}\rho^2 \sin\varphi \Big|_0^3 = \frac{9}{2}\sin\varphi.$$

Now the middle integral is

$$\int_0^{\pi} \frac{9}{2}\sin\varphi \, d\varphi = -\frac{9}{2}\cos\varphi \Big|_0^{\pi} = -\frac{9}{2}(\cos\pi - \cos 0) = -\frac{9}{2}(-1-1) = 9.$$

Finally, we have

$$\text{Mass} = \int_0^{2\pi} 9 \, d\theta = \boxed{18\pi}.$$

7. [14 points] Find $\int_{\mathcal{C}} (x-1)^2 + y^2 ds$, where $\mathcal{C}$ is the circle $x^2 + y^2 = 4$ (traversed once).

Solution: We parameterize $\mathcal{C}$ using $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix}$, for $0 \le t \le 2\pi$. Note that $\|\vec{\boldsymbol{x}}'(t)\| = 2$. Therefore,

$$\begin{aligned} \int_{\mathcal{C}} (x-1)^2 + y^2 ds &= \int_0^{2\pi} \left[(2\cos t - 1)^2 + (2\sin t)^2\right] \|\vec{\boldsymbol{x}}'(t)\| \, dt \\ &= 2\int_0^{2\pi} 4\cos^2 t - 4\cos t + 1 + 4\sin^2 t \, dt = 2\int_0^{2\pi} 5 - 4\cos t \, dt \\ &= 2(5t - 4\sin t)\Big|_0^{2\pi} = \boxed{20\pi}. \end{aligned}$$