Math 234	Exam I (with answers)	Spring 2017

1. Consider the points $A = (-2, 1, 1)$, $B = (1, -2, 1)$, and $C = (-3, 3, 3)$.

   (a) [8 points] Find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution: We will use the fact that $\overrightarrow{AB} \cdot \overrightarrow{AC} = \|\overrightarrow{AB}\|\|\overrightarrow{AC}\| \cos\theta$, where θ is the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Note that

$$\overrightarrow{AB} = \begin{pmatrix} 1-(-2) \\ -2-1 \\ 1-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix} -3-(-2) \\ 3-1 \\ 3-1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, \text{ so}$$

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{AC} &= 3 \cdot (-1) + (-3) \cdot 2 + 0 \cdot 2 = -9, \\ \|\overrightarrow{AB}\| &= \sqrt{3^2 + (-3)^2 + 0^2} = 3\sqrt{2}, \text{ and} \\ \|\overrightarrow{AC}\| &= \sqrt{(-1)^2 + 2^2 + 2^2} = 3. \end{aligned}$$

Therefore,

$$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|} = \frac{-9}{3\sqrt{2} \cdot 3} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

So the angle is $\theta = \arccos(-\sqrt{2}/2) = \dfrac{3\pi}{4}$.

   (b) [8 points] Find the area of the triangle formed by A, B, and C.

Solution: One way to solve this problem is to use the fact that $\|\overrightarrow{AB} \times \overrightarrow{AC}\|$ is the area of the parallelogram given by $\overrightarrow{AB}$ and $\overrightarrow{AC}$, so it is twice the area of the triangle. Now note that

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{vmatrix} \vec{\boldsymbol{i}} & 3 & -1 \\ \vec{\boldsymbol{j}} & -3 & 2 \\ \vec{\boldsymbol{k}} & 0 & 2 \end{vmatrix} = -6\vec{\boldsymbol{i}} - 6\vec{\boldsymbol{j}} + 3\vec{\boldsymbol{k}} = \begin{pmatrix} -6 \\ -6 \\ 3 \end{pmatrix}.$$

So the area of the triangle is $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{(-6)^2 + (-6)^2 + 3^2} = 9/2$.

   (c) [8 points] Find an equation for the plane containing A, B, and C.

Solution: The vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is normal to the plane containing A, B, and C. So the equation will look like

$$-6x - 6y + 3z = k,$$

for some constant k. To find k, we can plug in one of the points, say A:

$$-6 \cdot (-2) - 6 \cdot 1 + 3 \cdot 1 = 9 = k,$$

so the equation we are looking for is

$$\begin{aligned} -6x - 6y + 3z &= 9, \text{ or dividing by } -3, \\ 2x + 2y - z &= -3. \end{aligned}$$

(Note that it is very easy to check our work; just plug in the other two points, B and C, to see that they also satisfy this equation.)

2. Consider the vector function $\vec{\boldsymbol{x}}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t^2/2 \end{pmatrix}$.

(a) [8 points] Set up *but do not evaluate* an integral for the length of the curve parameterized by $\vec{\boldsymbol{x}}(t)$ from $t = 0$ to $t = \pi$.

Solution: First note that $\vec{\boldsymbol{x}}'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ t \end{pmatrix}$, so the speed of the parameterization is

$$\|\vec{\boldsymbol{x}}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + t^2} = \sqrt{1 + t^2}.$$

Therefore, the length from $t = 0$ to $t = \pi$ is

$$\int_0^\pi \|\vec{\boldsymbol{x}}'(t)\| \, dt = \int_0^\pi \sqrt{1 + t^2} \, dt.$$

(b) [8 points] Find the unit tangent vector $\vec{\boldsymbol{T}}(t)$ to the curve parameterized by $\vec{\boldsymbol{x}}(t)$.

Solution:

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{x}}'(t)}{\|\vec{\boldsymbol{x}}'(t)\|} = \frac{1}{\sqrt{1 + t^2}} \begin{pmatrix} -\sin t \\ \cos t \\ t \end{pmatrix}.$$

(c) [8 points] Find the curvature vector $\vec{\boldsymbol{\kappa}}(t)$ when $t = \pi$.

Solution:

$$\vec{\boldsymbol{\kappa}}(t) = \frac{d\vec{\boldsymbol{T}}(t)}{ds} = \frac{1}{\|\vec{\boldsymbol{x}}'(t)\|}\frac{d\vec{\boldsymbol{T}}(t)}{dt} = \frac{1}{\sqrt{1+t^2}}\frac{1}{dt}\left(\frac{1}{\sqrt{1+t^2}}\begin{pmatrix}-\sin t\\ \cos t\\ t\end{pmatrix}\right)$$
$$= \frac{1}{\sqrt{1+t^2}}\left(-\frac{t}{(1+t^2)^{3/2}}\begin{pmatrix}-\sin t\\ \cos t\\ t\end{pmatrix} + \frac{1}{\sqrt{1+t^2}}\begin{pmatrix}-\cos t\\ -\sin t\\ 1\end{pmatrix}\right)$$
$$= -\frac{t}{(1+t^2)^2}\begin{pmatrix}-\sin t\\ \cos t\\ t\end{pmatrix} + \frac{1}{1+t^2}\begin{pmatrix}-\cos t\\ -\sin t\\ 1\end{pmatrix}.$$

Therefore,

$$\vec{\boldsymbol{\kappa}}(\pi) = -\frac{\pi}{(1+\pi^2)^2}\begin{pmatrix}0\\ -1\\ \pi\end{pmatrix} + \frac{1}{1+\pi^2}\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$$
$$= \frac{1}{(1+\pi^2)^2}\begin{pmatrix}1+\pi^2\\ \pi\\ 1\end{pmatrix}.$$

3. (a) [8 points] For which values of the constant C is $f(x, y) = x^2 + 6xy + Cy^2$ positive definite?

Solution: Completing the square,

$$f(x, y) = x^2 + 6xy + Cy^2 = (x + 3y)^2 - (3y)^2 + Cy^2 = (x + 3y)^2 + (C - 9)y^2.$$

We see that $f(x, y)$ is positive definite as long as $C - 9 > 0$, i.e., when $C > 9$.

(b) [8 points] Consider the quadratic form $g(x, y) = 2x^2 - 8xy - 10y^2$.

- Classify it as positive definite, negative definite, semidefinite, or indefinite.
- Draw the zero set and indicate where g is positive and where it is negative.

Solution: Completing the square,

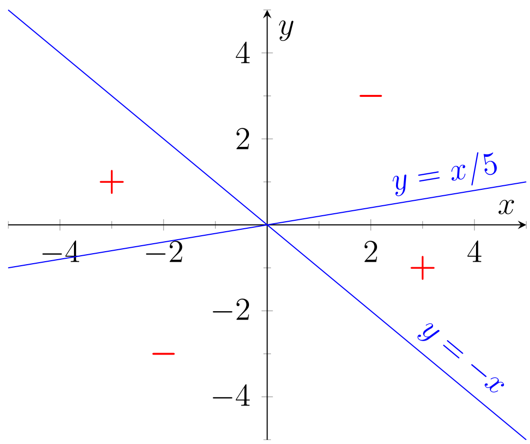
$$g(x, y) = 2\left[x^2 - 4xy - 5y^2\right] = 2\left[(x - 2y)^2 - (-2y)^2 - 5y^2\right]$$
$$= 2\left[(x - 2y)^2 - 9y^2\right].$$

So we see that $g(x, y)$ is an **indefinite** quadratic form. Using the fact that

$A^2 - B^2 = (A+B)(A-B)$, we have

$$g(x,y) = 2\left[(x-2y)^2 - 9y^2\right] = 2\left[(x-2y)^2 - (3y)^2\right]$$
$$= 2(x-2y+3y)(x-2y-3y) = 2(x+y)(x-5y).$$

Now we can see that $g(x,y) = 0$ when $x+y=0$ and when $x-5y=0$ giving us the two lines that make up the zero set. Using the fact that $g(1,0) = 2 > 0$, we can determine where g is positive and where it is negative.



4. Consider the function $f(x,y) = \dfrac{x^2+1}{x+y}$.

(a) [8 points] Find an equation for the tangent plane to $f(x,y)$ at the point $(2,3,1)$.

Solution: Using the quotient rule,

$$f_x(x,y) = \frac{2x(x+y) - (x^2+1)1}{(x+y)^2} = \frac{x^2+2xy-1}{(x+y)^2}, \text{ and}$$
$$f_y(x,y) = \frac{0(x+y) - (x^2+1)1}{(x+y)^2} = \frac{-x^2-1}{(x+y)^2}.$$

So $f_x(2,3) = 3/5$, and $f_y(2,3) = -1/5$. The equation for the tangent plane at $(2,3,1)$ is

$$z = f(2,3) + f_x(2,3)(x-2) + f_y(2,3)(y-3).$$

This becomes

$$z = 1 + \frac{3}{5}(x-2) - \frac{1}{5}(y-3).$$

Simplifying further, we get

$$3x - y - 5z = -2.$$

(b) [6 points] Find a vector normal to the graph of $z = f(x,y)$ at $(2,3,1)$.

Solution: The normal vector $\begin{pmatrix} 3 \\ -1 \\ -5 \end{pmatrix}$ can be read off from the coefficients in the equation of the tangent plane. Any nonzero multiple of this vector would also work, for example $\begin{pmatrix} 3/5 \\ -1/5 \\ -1 \end{pmatrix}$; this choice makes the relationship to the gradient more clear.

(c) [6 points] Find a vector normal to the level set $f^{-1}(1)$ at the point $(2, 3)$.

Solution: The gradient is normal to the level set. So we take

$$\vec{\nabla} f(2,3) = \begin{pmatrix} f_x(2,3) \\ f_y(2,3) \end{pmatrix} = \begin{pmatrix} 3/5 \\ -1/5 \end{pmatrix}.$$

5. Suppose that $f(x, y)$ is a function such that $f_x(x, y) = xy^2 + x$ and $f_y(x, y) = x^2 y$.

(a) [8 points] Consider the parameterization of the unit circle given by $x(t) = \cos t$ and $y(t) = \sin t$. Find $\dfrac{df}{dt}$ as a function of t.

Solution: The chain rule tells us that

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Filling in, we have

$$\begin{aligned} \frac{df}{dt} &= f_x(\cos t, \sin t)(-\sin t) + f_y(\cos t, \sin t)\cos t \\ &= (\cos t \sin^2 t + \cos t)(-\sin t) + (\cos^2 t \sin t)\cos t. \end{aligned}$$

Further simplification is not needed, but we could get it down to

$$\frac{df}{dt} = -2\cos t \sin^3 t.$$

(b) [8 points] Let $g(u, v) = f(u^2, uv)$. Find $\dfrac{\partial g}{\partial u}(u, v)$.

Solution: By the chain rule,

$$
\begin{aligned}
\frac{\partial g}{\partial u}(u,v) = \frac{\partial}{\partial u} f(u^2, uv) &= \frac{\partial f}{\partial x}\left(\frac{\partial}{\partial u}u^2\right) + \frac{\partial f}{\partial y}\left(\frac{\partial}{\partial u}uv\right) \\
&= f_x(u^2, uv) \cdot (2u) + f_y(u^2, uv) \cdot v \\
&= \left[u^2(uv)^2 + u^2\right] \cdot (2u) + \left[(u^2)^2 uv\right] \cdot v \\
&= 3u^5v^2 + 2u^3.
\end{aligned}
$$