Answers to Exercises for André's FRG Tutorials

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September 15, 2009

1. a.
Clearly $\hat{c}(x)$ is nonincreasing and uniformly left-c.e. Given such an f, there exists d computable and nondecreasing in t with $f(y) = \lim_t d(y,t)$. Let $c(x,s) = \max_{x \leq y, t < s} d(y,t)$, which is clearly monotonic. Note $\hat{c} = f$ since $\forall x$ $f(x) = \sup_{y \geq x} f(y) = \sup_{y,t \geq x} d(y,t) = \sup_s c(x,s) \geq \sup_t d(x,t) = f(x)$.

b.
Define computable functions f, g by

$$
\begin{aligned}
f(0) &= g(0) = 1 \\
f(2n+1) &= f(2n) & g(2n+1) &= f(2n+1)/n+1 \\
g(2n+2) &= g(2n+1) & f(2n+2) &= g(2n+2)/n+1.
\end{aligned}
$$

Observe $g(2n+1) \leq f(2n) \leq g(2n)$ and $f(2n+2) \leq g(2n+1) \leq f(2n+1)$. That is, f and g are nonincreasing, so we may apply a., to define cost functions c, d with $\hat{c} = f$, $\hat{d} = g$. Since for all n, $nf(2n+2) < g(2n+2)$, and $ng(2n+1) < f(2n+1)$, by Thm. 20 of the tutorial, $c \not\twoheadrightarrow d$ and $d \not\twoheadrightarrow c$.

2.
The existence of c follows from SJT $= (\mathcal{H})^\Diamond$ for $\mathcal{H}$ a null Σ^0_3 class (i.e., superlow) – use $c_{\mathcal{H}}$. Such a c cannot be benign because no single benign cost function characterizes SJT.

3.
By relativizing the universal ML-test to $\emptyset'$, there is a $\Pi^{0,\emptyset'}_1$ class P of 2-random reals. The relativized Low Basis Theorem yields $Z \in P$ with $(Z \oplus \emptyset')' \equiv_T \emptyset''$. Since Z is 2-random, by a result of Kautz $Z' \equiv_T Z \oplus \emptyset'$. Thus Z is Low_2.

4.
Since all K-trivial degrees are Δ^0_2, every noncomputable K-trivial degree is hyperimmune, and thus contains a weakly 1-generic real by a result of Kurtz. A weakly 1-generic real is contained in every open c.e. set of measure 1 since such sets are dense in 2^ω, and is thus weakly random. That is, every noncomputable K-trivial degree contains a weakly random set. However, every Schnorr random set is high or ML-random, and is therefore not K-trivial (recall all K-trivial sets are low).

5. The desired set may be enumerated in stages $\alpha < \omega_1^{CK}$. We assume the graph of $\underline{\mathbb{U}}$ is enumerated one element at a time and only at successor stages. Hence, for each α, there is at most one w where $\underline{K}_\alpha(w) \neq \underline{K}_{\alpha+1}(w)$. We will use the cost function

$$c(x, \alpha) = \sum_{w \geq x} 2^{-\underline{K}_\alpha(w)}$$

because when an x is enumerated, it may require new requests for infinitely many initial segments. Since $c(x, \alpha)$ need not be in $\mathbb{Q}_2$, we use $Q(x, \alpha) := \{q \in \mathbb{Q}_2 | q < c(x, \alpha)\}$. To determine $Q(x, \alpha+1)$ effectively from $Q(x, \alpha)$, if $2^{-\underline{K}_\alpha(w)}$ increases by r for a (unique) $w \geq x$, we add to $Q(x, \alpha)$ $q + r$ for all $q \in Q(x, \alpha)$ and all $q < r$.

The construction proceeds as in the construction of a simple K-trivial, save using $Q(x, \alpha)$ to determine if the cost of an enumeration is low enough. Using the uniform enumeration of Π^1_1 sets, the resulting real intersects all infinite Π^1_1 sets and is not Δ^1_1. That the resulting real is $\underline{K}$-trivial is proved in the same way as before, except by building a bounded Π^1_1 request set.

Additional details for 5 (originally omitted to avoid exceeding 15 line limit):

Let A be the real constructed above, and assume the cost of enumerating A is at most 1. We prove A is $\underline{K}$-trivial by first enumerating a Π^1_1 bounded request set W. Requests of the form $\langle \underline{K}_{\alpha+1}(w)+1, A_{\alpha+1} \upharpoonright w\rangle$ are enumerated into W at stage $\alpha+1$ if $\underline{K}_{\alpha+1}(w) \neq \underline{K}_\alpha(w)$, or if both $\underline{K}_{\alpha+1}(w) < \infty$ and $A_{\alpha+1} \upharpoonright w \neq A_\alpha \upharpoonright w$ (although Π^1_1 enumeration was given by enumerating at most 1 element per stage, as with computable enumeration it is clear more than 1, or even infinitely many elements may be enumerated in a stage, as long as the enumeration is uniform). W is bounded, because requests of the first kind contribute at most $\underline{\Omega}/2$ and requests of the second kind contribute at most 1/2 (since $2^{-\underline{K}_{\alpha+1}(w)}$ is in $Q(x, \alpha)$, where x is the element (not more than w) enumerated in A at stage $\alpha + 1$).

Let M_d be a Π^1_1 machine for W. We claim $\underline{K}(A \upharpoonright w) \leq \underline{K}(w) + d + 1$ for all w. Given w, let α be maximal such that $\alpha = 0$ or $A_{\alpha+1} \upharpoonright w \neq A_\alpha \upharpoonright w$. In the latter case,

$$\underline{K}(A \upharpoonright w) \leq \underline{K}_{M_d}(A \upharpoonright w) + d \leq \underline{K}_{\alpha+1}(w) + 1 + d,$$

caused by a request at stage $\alpha+1$. In this case, if $\underline{K}_{\alpha+1}(w) = \underline{K}(w)$, the claim is proven. If $\underline{K}_{\alpha+1}(w) \neq \underline{K}(w)$, or $\alpha = 0$, then the inequality is caused by a request at stage $\beta + 1$, where $\beta > \alpha$ is minimal such that $\underline{K}_{\beta+1}(w) = \underline{K}(w)$.

6. We wish to carry out the Friedburg-Mucnik construction with Π^1_1 enumerations and fin-h reductions in place of computable enumerations and Turing computations. To do this, we need to index the fin-h reductions, which we can if we remove the condition that $\mathrm{dom}(\Phi)$ be closed under prefixes – simply enumerate all partial functions with Π^1_1 graph, and ignore any enumeration that would violate the compatibility rule. The resulting index Φ_e of reductions will include all the fin-h reductions, but maybe also some others. We seek to fulfill the requirements

$$R_{2e} : P \neq \Phi^Q_e$$
$$R_{2e+1} : Q \neq \Phi^P_e$$

We use the same method as in the original construction – i.e., decide whether or not to enumerate x into $P^{[e]}$ if $\Phi^{Q_{\alpha+1}}_{e,\alpha+1}(x)$ converges, making sure to only use x if it is larger than the uses of all lower-numbered requirements that are marked as satisfied. The construction is run over stages $\alpha < \omega^{CK}_1$. The verification that this works proceeds exactly as it does in the computable case.

7.
Since $\underline{\Omega}$ is left-Π^1_1, $\{\underline{\Omega}\}$ is a Π^1_1 class, as $X = \underline{\Omega}$ iff

$$\forall Z[\forall n(X \upharpoonright n \leq Z \upharpoonright n) \vee \exists q(q < X \wedge Z < q)],$$

where q ranges over $\mathbb{Q}_2$. But $\{\underline{\Omega}\}$ is not Δ^1_1 since $\underline{\Omega}$ is Δ^1_1-random.