Randomness via effective descriptive set theory

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- A formal randomness notion is introduced by specifying a test concept.
- Usually the null classes given by tests are arithmetical.
- Here we give formal definitions of randomness notions using tools from higher computability theory.

1 Introduction

1.1 Informal introduction to Π^1_1 relations

Let $2^{\mathbb{N}}$ denote Cantor space.

- A relation $\mathcal{B} \subseteq \mathbb{N}^k \times (2^{\mathbb{N}})^r$ is Π^1_1 if it is obtained from an arithmetical relation by a universal quantification over sets.
- If $k = 1, r = 0$ we have a Π^1_1 *set* $\subseteq \mathbb{N}$.
- If $k = 0, r = 1$ we have a Π^1_1 *class* $\subseteq 2^{\mathbb{N}}$.
- The relation $\mathcal{B}$ is Δ^1_1 if both $\mathcal{B}$ and its complement are Π^1_1.

There is an equivalent representation of Π^1_1 relations where the members are enumerated at stages that are countable ordinals. For Π^1_1 sets (of natural numbers) these stages are in fact computable ordinals, i.e., the order types of computable well-orders.

1.2 New closure properties

- Analogs of many notions from the computability setting exist in the setting of higher computability, but...
- the results about them often turn out to be different.

The reason is that there are two new closure properties.

(C1) The Π^1_1 and Δ^1_1 relations are closed under number quantification.

(C2) If a function f maps each number n in a certain effective way to a computable ordinal, then the range of f is bounded by a computable ordinal. This is the Bounding Principle.

1.3 Further notions

Beyond the Π^1_1 version of ML-randomness, we will study Δ^1_1-randomness and Π^1_1-randomness. The tests are simply the null Δ^1_1 classes and the null Π^1_1 classes, respectively.

The implications are

$$\Pi^1_1\text{-}\textit{randomness} \Rightarrow \Pi^1_1\text{-}\textit{ML-randomness} \Rightarrow \Delta^1_1\text{-}\textit{randomness.}$$

The converse implications fail.

1.4 The story

- Martin-Löf (1970) was the first to study randomness in the setting of higher computability theory.
- Surprisingly, he suggested Δ^1_1-randomness as the appropriate mathematical concept of randomness.
- His main result was that the union of all Δ^1_1 null classes is a Π^1_1 class that is not Δ^1_1.
- Later it turned out that Δ^1_1-randomness is the higher analog of *both* Schnorr and computable randomness.

1.5 Limits of effectivity

- The strongest notion we will consider is Π^1_1-randomness, which has no analog in the setting of computability theory.
- This is where we reach the limits of effectivity.
- Interestingly, there is a *universal test*, in this case, a largest Π^1_1 null class.

2 Preliminaries on higher computability theory

- We give more details on Π^1_1 and Δ^1_1 relations.
- We formulate a few principles in effective DST from which most results can be obtained. They are proved in Sacks.

2.1 Arithmetical relations

Let $\mathcal{A} \subseteq \mathbb{N}^k \times 2^{\mathbb{N}}$ and $n \geq 1$. $\mathcal{A}$ is Σ^0_n if

$$\langle e_1, \ldots, e_k, X \rangle \in \mathcal{A} \leftrightarrow \exists y_1 \forall y_2 \ldots Q y_n \, R(e_1, \ldots, e_k, y_1, \ldots, y_{n-1}, X\restriction_{y_n}),$$

where R is a computable relation, and Q is "$\exists$" if n is odd and Q is "$\forall$" if n is even.
$\mathcal{A}$ is *arithmetical* if $\mathcal{A}$ is Σ^0_n for some n.
We can also apply this to relations $\mathcal{A} \subseteq \mathbb{N}^k \times (2^{\mathbb{N}})^n$, replacing a tuple of sets $X_1, \ldots, X_n$ by the single set $X_1 \oplus \ldots \oplus X_n$.

2.2 Π^1_1 and other relations

Definition 2.1. *Let $k, r \geq 0$ and $\mathcal{B} \subseteq \mathbb{N}^k \times (2^{\mathbb{N}})^r$. $\mathcal{B}$ is Π^1_1 if there is an arithmetical relation $\mathcal{A} \subseteq \mathbb{N}^k \times (2^{\mathbb{N}})^{r+1}$ such that*
$\langle e_1, \ldots, e_k, X_1, \ldots, X_r \rangle \in \mathcal{B} \leftrightarrow$
$$\forall Y \, \langle e_1, \ldots, e_k, X_1, \ldots, X_r, Y \rangle \in \mathcal{A}.$$
$\mathcal{B}$ is Σ^1_1 if its complement is Π^1_1, and $\mathcal{B}$ is Δ^1_1 if it is both Π^1_1 and Σ^1_1. A Δ^1_1 set is also called hyperarithmetical*.*

- The Π^1_1 relations are closed under the application of number quantifiers.
- So are the Σ^1_1 and Δ^1_1 relations.
- One can get away with $\mathcal{A}$ in Σ^0_2.

2.3 Well-orders and computable ordinals

In the following we will consider binary relations $W \subseteq \mathbb{N} \times \mathbb{N}$ with domain an initial segment of $\mathbb{N}$. They can be encoded by sets $R \subseteq \mathbb{N}$ via the usual pairing function. We identify the relation with its code.
A linear order R is a well-order if each non-empty subset of its domain has a least element. The class of well-orders is Π^1_1. Furthermore, the index set $\{e\colon\ W_e \text{ is a well-order}\}$ is Π^1_1.
Given a well-order R and an ordinal α, we let $|R|$ denote the order type of R, namely, the ordinal α such that $(\alpha, \in)$ is isomorphic to R.
We say that α is *computable* if $\alpha = |R|$ for a computable well-order R. Each initial segment of a computable well-order is also computable. So the computable ordinals are closed downwards.

2.4 Lowness for ω_1^{ck}

We let ω_1^Y denote the least ordinal that is not computable in Y. The least incomputable ordinal is ω_1^{ck} (which equals $\omega_1^{\emptyset}$).

An important example of a Π^1_1 class is

$$\mathcal{C} = \{Y \colon \omega_1^Y > \omega_1^{\mathrm{ck}}\}.$$

(To see that this class is Π^1_1, note that $Y \in \mathcal{C} \leftrightarrow \exists e \left[\Phi_e^Y \text{ is well-order} \,\&\, \forall i \; [W_i \text{ is computable relation} \rightarrow \Phi_e^Y \not\cong W_i]\right]$. This can be put into Π^1_1 form because the Π^1_1 relations are closed under number quantification.)

If $\omega_1^Y = \omega_1^{\mathrm{ck}}$ we say that Y is *low for* ω_1^{ck}.

2.5 Representing Π^1_1 relations by well-orders

- A Σ^0_1 class, of the form $\{X \colon \exists y \, R(X \restriction_y)\}$ for computable R, can be thought of as being enumerated at stages $y \in \mathbb{N}$.
- Π^1_1 classes can be described by a generalized type of enumeration where the stages are countable ordinals.

Theorem 2.2 (Representing Π^1_1 relations)**.** *Let $k, r \geq 0$. Given a Π^1_1 relation $\mathcal{B} \subseteq \mathbb{N}^k \times (2^{\mathbb{N}})^r$, there is a computable function $p \colon \mathbb{N}^k \rightarrow \mathbb{N}$ such that*

$$\langle e_1, \ldots, e_k, X_1 \oplus \ldots \oplus X_r \rangle \in \mathcal{B} \leftrightarrow \Phi_{p(e_1,\ldots,e_k)}^{X_1 \oplus \ldots \oplus X_r} \text{ is a well-order.}$$

Conversely, each relation given by such an expression is Π^1_1.

The order type of $\Phi_{p(e_1,\ldots,e_k)}^{X_1 \oplus \ldots \oplus X_r}$ is the stage at which the element enters $\mathcal{B}$, so for a countable ordinal α, we let

$$\mathcal{B}_\alpha = \{\langle e_1, \ldots, e_k, X_1 \oplus \ldots \oplus X_r \rangle \colon |\Phi_{p(e_1,\ldots,e_k)}^{X_1 \oplus \ldots \oplus X_r}| < \alpha\}.$$

Thus, $\mathcal{B}_\alpha$ contains the elements that enter $\mathcal{B}$ before stage α.

2.6 A Π^1_1 complete set, and indices for Π^1_1 relations

Recall we may view sets as relations $\subseteq \mathbb{N} \times \mathbb{N}$. By the above,

$$O = \{e \colon W_e \text{ is a well-order}\}$$

is a Π^1_1-*complete set*. That is, O is Π^1_1 and $S \leq_m O$ for each Π^1_1 set S.

For $p \in \mathbb{N}$, we let $\mathcal{Q}_p$ denote the Π^1_1 *class* with index p. Thus,

$$\mathcal{Q}_p = \{X \colon \Phi_p^X \text{ is a well-order}\}.$$

Note that $\mathcal{Q}_{p,\alpha} = \{X \colon |\Phi_p^X| < \alpha\}$, so that $X \in \mathcal{Q}_p$ implies $X \in \mathcal{Q}_{p,|\Phi_p^X|+1}$.

2.7 Relativization

The notions introduced above can be relativized to a set A. It suffices to include A as a further set variable in definition of Π^1_1 relations. For instance, $S \subseteq \mathbb{N}$ is a $\Pi^1_1(A)$ set if $S = \{e\colon \langle e, A\rangle \in \mathcal{B}\}$ for a Π^1_1 relation $\mathcal{B} \subseteq \mathbb{N} \times 2^{\mathbb{N}}$.
The following set is $\Pi^1_1(A)$-complete:

$$O^A = \{e\colon W^A_e \text{ is a well-order}\}.$$

A Π^1_1 object can be approximated by Δ^1_1 objects.

Lemma 2.3 (Approximation Lemma)**.** *(i) For each Π^1_1 set S and each $\alpha < \omega_1^{\mathrm{ck}}$, the set S_α is Δ^1_1.*
(ii) For each Π^1_1 class $\mathcal{B}$ and each countable ordinal α, the class $\mathcal{B}_\alpha$ is $\Delta^1_1(R)$, for every well-order R such that $|R| = \alpha$.

2.8 Π^1_1 classes and the uniform measure

Theorem 2.4 (Lusin)**.** *Each Π^1_1 class is measurable.*

The following frequently used result states that the measure of a class has the same descriptive complexity as the class itself. Note that (ii) follows from (i).

Lemma 2.5 (Measure Lemma)**.** *(i) For each Π^1_1 class, the real $\lambda\mathcal{B}$ is left-Π^1_1.*

(ii) If $\mathcal{S}$ is a Δ^1_1 class then the real $\lambda\mathcal{S}$ is left-Δ^1_1.

Theorem 2.6 (Sacks-Tanaka)**.** *A Π^1_1 class that is not null has a hyperarithmetical member.*

2.9 Reducibilities

Turing reducibility has two analogs in the new setting.
(1) Intuitively, as the stages are now countable ordinals, it is possible to look at the whole oracle set during a "computation". If full access to the oracle set is granted we obtain hyperarithmetical reducibility: $X \leq_h A$ iff $X \in \Delta^1_1(A)$.
(2) If only a finite initial segment of the oracle can be used we have the restricted version $\leq_{\textit{fin-h}}$.

2.10 Measure and reducibilities

Theorem 2.7 (Sacks 69)**.** *$A \notin \Delta^1_1 \Leftrightarrow \{X\colon X \geq_h A\}$ is null.*

Next, we reconsider the class of sets that are not low for ω_1^{ck}.

Theorem 2.8 (Spector 55)**.** *$O \leq_h X \Leftrightarrow \omega_1^{\mathrm{ck}} < \omega_1^X$.*

The foregoing two theorems yield:

Corollary 2.9. *The Π^1_1 class $\mathcal{C} = \{Y\colon \omega_1^Y > \omega_1^{\mathrm{ck}}\}$ is null.*

2.11 The Gandy Basis Theorem

The following result is an analog of the Low Basis Theorem. The proof differs from the proof of LBT because Σ^1_1 classes are not closed in general.

Theorem 2.10 (Gandy Basis Theorem)**.** *Let $\mathcal{S} \subseteq 2^{\mathbb{N}}$ be a non-empty Σ^1_1 class. Then there is $A \in \mathcal{S}$ such that $A \leq_T O$ and $O^A \leq_h O$ (whence $A <_h O$).*

Definition 2.11. *A fin-h reduction procedure is a partial function $\Phi\colon \{0,1\}^* \to \{0,1\}^*$ with Π^1_1 graph such that $\mathrm{dom}(\Phi)$ is closed under prefixes and, if $\Phi(x)\downarrow$ and $y \preceq x$, then $\Phi(y) \preceq \Phi(x)$.*
We write $A = \Phi^Z$ if $\forall n \exists m\ \Phi(Z\upharpoonright_m) \succeq A\upharpoonright_n$, and
$A \leq_{\textit{fin-h}} Z$ if $A = \Phi^Z$ for some fin-h reduction procedure Φ.

If A is hyperarithmetical then $\Phi = \{\langle x, A\upharpoonright_{|x|}\rangle\colon\ x \in \{0,1\}^*\}$ is Π^1_1, so $A \leq_{\textit{fin-h}} Z$ via Φ for any Z.

2.12 A set theoretical view

For a set S we let

- $L(0, S)$ be the transitive closure of $\{S\} \cup S$.
- $L(\alpha + 1, S)$ contains the sets that are first-order definable with parameters in $(L(\alpha, S), \in)$, and
- $L(\eta, S) = \bigcup_{\alpha<\eta} L(\alpha, S)$ for a limit ordinal η.

We write $L(\alpha)$ for $L(\alpha, \emptyset)$.

A Δ_0 *formula* is a first-order formula in the language of set theory which involves only bounded quantification, namely, quantification of the form $\exists z \in y$ and $\forall z \in y$.

A Σ_1 *formula* has the form $\exists x_1 \exists x_2 ... \exists x_n\, \varphi_0$ where φ_0 is Δ_0.
By Theorem 2.2 we can view Π^1_1 sets as being enumerated at stages that are computable ordinals. The following important theorem provides a further view of this existential aspect of Π^1_1 sets.

Theorem 2.12 (Gandy/Spector, 55)**.** *$S \subseteq \mathbb{N}$ is Π^1_1 $\Leftrightarrow$ there is a Σ_1-formula $\varphi(y)$ such that*

$$S = \{y \in \omega\colon\ (L(\omega_1^{\mathrm{ck}}), \in) \models \varphi(y)\}.$$

Given $A \subseteq \mathbb{N}$, let $L_A = L(\omega_1^A, A)$. We say that $D \subseteq (L_A)^k$ is Σ_1 *over* L_A if there is a Σ_1 formula φ such that

$$D = \{\langle x_1, \ldots, x_k\rangle \in (L_A)^k\colon\ (L_A, \in) \models \varphi(x_1, \ldots, x_k)\}.$$

Thus, $S \subseteq \mathbb{N}$ is Π^1_1 iff S is Σ_1 over $L(\omega_1^{\mathrm{ck}})$.
We often consider partial functions from L_A to L_A with a graph defined by a Σ_1 formula with parameters. We say the function is Σ_1 over L_A. Such functions are an analog of functions partial computable in A.

Lemma 2.13 (Bounding Principle)**.** *Suppose $f\colon \omega \to \omega_1^A$ is Σ_1 over L_A. Then there is an ordinal $\alpha < \omega_1^A$ such that $f(n) < \alpha$ for each n.*

2.13 Summary of tools

Approximation Lemma 2.3. (i) For each Π^1_1 set S and each $\alpha < \omega_1^{\mathrm{ck}}$, the set S_α is Δ^1_1.
(ii) For each Π^1_1 class $\mathcal{B}$ and each countable ordinal α, the class $\mathcal{B}_\alpha$ is $\Delta^1_1(R)$, for every well-order R such that $|R| = \alpha$.

Measure Lemma 2.5. (i) For each Π^1_1 class, the real $\lambda\mathcal{B}$ is left-Π^1_1.
(ii) If $\mathcal{S}$ is a Δ^1_1 class then the real $\lambda\mathcal{S}$ is left-Δ^1_1.

Bounding Principle 2.13. Suppose $f\colon \omega \to \omega_1^A$ is Σ_1 over L_A. Then there is an ordinal $\alpha < \omega_1^A$ such that $f(n) < \alpha$ for each n.

3 Analogs of Martin-Löf randomness and K-triviality

We develop an analog of the theory of ML-randomness and K-triviality based on Π^1_1 sets. The definitions and results are due to Hjorth and Nies (2007)

3.1 Π^1_1 Machines and prefix-free complexity

Definition 3.1. *A* Π^1_1-machine *is a possibly partial function $M\colon \{0,1\}^* \to \{0,1\}^*$ with a Π^1_1 graph. For $\alpha \le \omega_1^{\mathrm{ck}}$ we let $M_\alpha(\sigma) = y$ if $\langle \sigma, y\rangle \in M_\alpha$.*
We say that M is prefix-free if $\mathrm{dom}(M)$ *is prefix-free.*

There is an effective listing $(M^d)_{d\in\mathbb{N}}$ of all the prefix-free Π^1_1-machines.
As a consequence, there is an optimal prefix-free Π^1_1-machine.

Definition 3.2. *The prefix-free Π^1_1-machine $\underline{\mathbb{U}}$ is given by $\underline{\mathbb{U}}(0^d1\sigma) \simeq M^d(\sigma)$. Let $\underline{K}(y) = \min\{|\sigma|\colon \underline{\mathbb{U}}(\sigma) = y\}$.*
For $\alpha \le \omega_1^{\mathrm{ck}}$ let $\underline{K}_\alpha(y) = \min\{|\sigma|\colon \underline{\mathbb{U}}_\alpha(\sigma) = y\}$.

Since $\underline{\mathbb{U}}$ has Π^1_1 graph, the relation "$\underline{K}(y) \le u$" is Π^1_1 and, by 2.3, for $\alpha < \omega_1^{\mathrm{ck}}$ the relation "$\underline{K}_\alpha(y) \le u$" is Δ^1_1. Moreover $\underline{K} \le_T O$.

3.2 Machine Existence Theorem

A Π^1_1 set $W \subseteq \mathbb{N}\times\{0,1\}^*$ is called a Π^1_1 *bounded request set* if $1 \ge \sum_\rho 2^{-(\rho)_0}[\![\rho \in W]\!]$. (Here $(\rho)_0$ is the first component r of the pair $\rho = \langle r, y\rangle$.)

Theorem 3.3 (Machine Existence)**.** *From a Π^1_1 bounded request set W one can effectively obtain a prefix-free Π^1_1-machine M such that*

$$\forall \langle r, y\rangle \in W\, \exists w\, [|w| = r \;\&\; M(w) = y].$$

3.3 A version of Martin-Löf randomness based on Π^1_1 sets

A Π^1_1-*ML-test* is a sequence $(G_m)_{m\in\mathbb{N}}$ of open sets such that $\forall m \in \mathbb{N}\ \lambda G_m \leq 2^{-m}$ and the relation $\{\langle m, \sigma\rangle \colon\ [\sigma] \subseteq G_m\}$ is Π^1_1. A set Z is Π^1_1-*ML-random* if $Z \notin \bigcap_m G_m$ for each Π^1_1-ML-test $(G_m)_{m\in\mathbb{N}}$.
For $b \in \mathbb{N}$ let $\underline{\mathcal{R}}_b = [\{x \in \{0,1\}^* \colon\ \underline{K}(x) \leq |x| - b\}]^{\prec}$.

Proposition 3.4. *$(\underline{\mathcal{R}}_b)_{b\in\mathbb{N}}$ is a Π^1_1-ML-test.*

We have a higher analog of the Levin-Schnorr Theorem:

$$Z \text{ is } \Pi^1_1\text{-ML-random} \ \Leftrightarrow\ Z \in 2^{\mathbb{N}} - \underline{\mathcal{R}}_b \text{ for some } b.$$

Since $\bigcap_b \underline{\mathcal{R}}_b$ is Π^1_1, this implies that the class of Π^1_1-ML-random sets is Σ^1_1.
We provide two examples of Π^1_1-ML-random sets.

1. By the Gandy Basis Theorem there is a Π^1_1-ML-random set $Z \leq_T O$ such that $O^Z \leq_h O$.
2. Let $\underline{\Omega} = \lambda[\text{dom}\,\underline{\mathbb{U}}]^{\prec} = \sum_\sigma 2^{-|\sigma|}\, [\![\underline{\mathbb{U}}(\sigma)\downarrow]\!]$. Note that $\underline{\Omega}$ is left-Π^1_1. $\underline{\Omega}$ is shown to be Π^1_1-ML-random similar to the usual proof.

Theorem 3.5 (Kučera - Gács)**.** *Let Q be a closed Σ^1_1 class of Π^1_1-ML-random sets such that $\lambda Q \geq 1/2$ (say $Q = 2^{\mathbb{N}} - \underline{\mathcal{R}}_1$). For each set A there is $Z \in Q$ such that $A \leq_{\textit{fin-h}} Z$.*

3.4 An analog of K-triviality

Definition 3.6. *A is $\underline{K}$-trivial if $\exists b \forall n\, \underline{K}(A\restriction_n) \leq \underline{K}(n) + b$.*
*Given a limit ordinal $\eta \leq \omega_1^{\mathrm{ck}}$, we say that A is $\underline{K}$-*trivial at η *if $\exists b \forall n\, \underline{K}_\eta(A\restriction_n) \leq \underline{K}_\eta(n) + b$.*

Proposition 3.7. *There is a $\underline{K}$-trivial Π^1_1 set A that is not Δ^1_1.*

To prove this, one transfers the usual cost function construction to the setting of Π^1_1 sets.
Fix b and $\eta \leq \omega_1^{\mathrm{ck}}$. The sets that are $\underline{K}$-trivial via b at η are the paths through the tree

$$T_{\eta,b} = \{z \colon\ \forall u \leq |z|\ \underline{K}_\eta(z\restriction_u) \leq \underline{K}_\eta(u) + b\}.$$

If $\eta < \omega_1^{\mathrm{ck}}$ then $T_{\eta,b}$ is Δ^1_1 because $\underline{\mathbb{U}}_\eta$ is a Δ^1_1 set.

Proposition 3.8 (Higher Chaitin)**.** *Let $\eta \leq \omega_1^{\mathrm{ck}}$ be a limit ordinal (with the technical closure condition $\forall \alpha < \eta[g(\alpha) < \eta$ for a certain Δ^1_1 order function g).*

- *For each b, at most $O(2^b)$ sets are $\underline{K}$-trivial at η with constant b.*
- *If $\eta < \omega_1^{\mathrm{ck}}$ and A is $\underline{K}$-trivial at η, then A is hyperarithmetical.*

Lemma 3.9. *If A is $\underline{K}$-trivial and $\omega_1^A = \omega_1^{\mathrm{ck}}$, then A is hyperarithmetical.*

Proof. Suppose A is $\underline{K}$-trivial via b. We show that A is $\underline{K}$-trivial at η via b for some appropriate $\eta < \omega_1^{\mathrm{ck}}$.
Recall that $L_A = L(\omega_1^A, A)$. We define by transfinite recursion in L_A a function $h\colon \omega \to \omega_1^{\mathrm{ck}}$ which is Σ_1 over L_A: let $h(0) = 0$ and

$$h(n+1) = \mu\beta > g(h(n)).\, \forall m \le n\, \underline{K}_\beta(A\restriction_m) \le \underline{K}_\beta(m) + b.$$

Since A is $\underline{K}$-trivial, $h(n)$ is defined for each $n \in \omega$. Let $\eta = \sup(\mathrm{ran}\, h)$. Tthen $\eta < \omega_1^A = \omega_1^{\mathrm{ck}}$ by the Bounding Principle, so η is as required.

3.5 Lowness for Π^1_1-ML-randomness

$\underline{\mathsf{MLR}}^A$ denotes the class of sets which are Π^1_1-ML-random relative to A.

Definition 3.10. *A is* low for Π^1_1-ML-randomness *if* $\underline{\mathsf{MLR}}^A = \underline{\mathsf{MLR}}$.
A is called a base for Π^1_1-ML-randomness *if $A \le_{\mathit{fin\text{-}h}} Z$ for some $Z \in \underline{\mathsf{MLR}}^A$.*

By the Π^1_1 version of the Kučera- GácsTheorem, a set that is low for Π^1_1-ML-randomness is a base for Π^1_1-ML-randomness.

Theorem 3.11. *A is a base for Π^1_1-ML-randomness $\Leftrightarrow$ A is Δ^1_1.*

$\Leftarrow$: $A \le_{\mathit{fin\text{-}h}} Z$ for each Z, so A is a base for Π^1_1-ML-randomness.
$\Rightarrow$: Suppose A is a base for Π^1_1-ML-randomness. Thus $A = \Phi^Z$ for some *fin-h* reduction procedure Φ and $Z \in \underline{\mathsf{MLR}}^A$.
Assume for a contradiction that A is not Δ^1_1.
Step 1. We show that $\omega_1^A = \omega_1^{\mathrm{ck}}$. The class $\{Y\colon\ A = \Phi^Y\}$ is null. For each n let

$$V_n = [\{\rho\colon\ A\restriction_n \preceq \Phi^\rho\}]^\prec = [\{\rho\colon\ \exists\alpha < \omega_1^{\mathrm{ck}}\, A\restriction_n \preceq \Phi^\rho_\alpha\}]^\prec.$$

If $\omega_1^{\mathrm{ck}} < \omega_1^A$ then by the Approximation Lemma 2.3(i) relative to A, the following set is $\Delta^1_1(A)$:

$$\{\langle n, \rho\rangle\colon\ \exists\alpha < \omega_1^{\mathrm{ck}}\, A\restriction_n \preceq \Phi^\rho_\alpha\}\ .$$

So by the Measure Lemma 2.5, the following function is $\Delta^1_1(A)$:

$$h(n) = \mu k.\, \lambda V_k \le 2^{-n}.$$

Then $(V_{h(n)})_{n\in\mathbb{N}}$ is a Π^1_1-ML-test relative to A which succeeds on Z, contrary to the hypothesis that $Z \in \underline{\mathsf{MLR}}^A$.
Step 2. We show that A is $\underline{K}$-trivial, which together with $\omega_1^A = \omega_1^{\mathrm{ck}}$ implies that A is Δ^1_1 (Lemma 3.9).

This is a straightforward adaptation of the result of Hirschfeldt, Nies, and Stephan (2007) that in the computability setting, each base for ML-randomness (defined in terms of $\le_T$) is K-trivial.

Note that $\le_T$ becomes $\le_{\mathit{fin\text{-}h}}$ here.

4 Δ^1_1-randomness and Π^1_1-randomness

- We show that Δ^1_1-randomness coincides with the higher analogs of both Schnorr randomness and computable randomness.
- There is a universal test for Π^1_1 randomness
- Z is Π^1_1-random $\Leftrightarrow$ Z is Δ^1_1-random and $\omega_1^Z = \omega_1^{\mathrm{ck}}$.

Definition 4.1. *Z is Δ^1_1-random if Z avoids each null Δ^1_1 class (Martin-Löf, 1970). Z is Π^1_1-random if Z avoids each null Π^1_1 class (Sacks, 1990).*

We have the proper implications

$$\Pi^1_1\text{-random} \Rightarrow \Pi^1_1\text{-}ML\text{-random} \Rightarrow \Delta^1_1\text{-random}$$

- Δ^1_1 random is equivalent to ML-random in each $\emptyset^{(\alpha)}$, where α is a computable ordinal.
- Each Π^1_1-random set Z satisfies $\omega_1^Z = \omega_1^{\mathrm{ck}}$ because the Π^1_1 class $\{X\colon\ \omega_1^X > \omega_1^{\mathrm{ck}}\}$ is null.
- Thus, the Π^1_1-ML-random set $\underline{\Omega}$ is not Π^1_1-random, because $\underline{\Omega} \equiv_{wtt} O$.

4.1 Notions that coincide with Δ^1_1-randomness

A Π^1_1-*Schnorr test* is a Π^1_1-ML-test $(G_m)_{m\in\mathbb{N}}$ such that λG_m is left-Δ^1_1 uniformly in m. A supermartingale $M\colon\ \{0,1\}^* \to \mathbb{R}^+ \cup \{0\}$ is *hyperarithmetical* if $\{\langle x, q\rangle :\ q \in \mathbb{Q}_2\ \&\ M(x) > q\}$ is Δ^1_1. Its success set is $\mathsf{Succ}(M) = \{Z\colon\ \limsup_n M(Z\upharpoonright_n) = \infty\}$.

Theorem 4.2. *(i) Let $\mathcal{A}$ be a null Δ^1_1 class. Then $\mathcal{A} \subseteq \bigcap G_m$ for some Π^1_1-Schnorr test $\{G_m\}_{m\in\mathbb{N}}$ such that $\lambda G_m = 2^{-m}$ for each m.*

(ii) If $(G_m)_{m\in\mathbb{N}}$ is a Π^1_1-Schnorr test then $\bigcap_m G_m \subseteq \mathsf{Succ}(M)$ for some hyperarithmetical martingale M.

(iii) $\mathsf{Succ}(M)$ is a null Δ^1_1 class for each hyperarithmetical supermartingale M.

The foregoing characterization of Δ^1_1-randomness via hyperarithmetical martingales can be used to separate it from Π^1_1-ML-randomness.

Theorem 4.3. *For every unbounded non-decreasing hyperarithmetical function h, there is a Δ^1_1-random set Z such that $\forall^\infty n\ \underline{K}(Z\upharpoonright_n \mid n) \le h(n)$.*

The higher analog of the Levin-Schnorr Theorem implies:

Corollary 4.4. *There is a Δ^1_1-random set that is not Π^1_1-ML-random.*

By Sacks-Tanaka the class of Δ^1_1-random sets is not Π^1_1. In particular, there is no largest null Δ^1_1 class. However, the class of Δ^1_1-random sets is Σ^1_1 (Martin-Löf; see Book, Ex 9.3.11).

4.2 More on Π^1_1-randomness

There is a universal test for Π^1_1-randomness.

Theorem 4.5 (Kechris (1975); Hjorth, Nies (2007)). *There is a null Π^1_1 class $\mathcal{Q}$ such that $\mathcal{S} \subseteq \mathcal{Q}$ for each null Π^1_1 class $\mathcal{S}$.*

Proof.

- We show that one may effectively determine from a Π^1_1 class $\mathcal{S}$ a null Π^1_1 class $\widehat{\mathcal{S}} \subseteq \mathcal{S}$ such that

$$\mathcal{S} \text{ is null} \Rightarrow \widehat{\mathcal{S}} = \mathcal{S}.$$

- Assuming this, let $\mathcal{Q}_p$ be the Π^1_1 class given by the Turing functional Φ_p in the sense of Theorem 2.2. Then $\mathcal{Q} = \bigcup_p \widehat{\mathcal{Q}}_p$ is Π^1_1, so $\mathcal{Q}$ is as required.

4.3 A Π^1_1-random Turing below O

Applying the Gandy Basis Theorem to the Σ^1_1 class $2^{\mathbb{N}} - \mathcal{Q}$ yields:

Corollary 4.6. *There is a Π^1_1-random set $Z \leq_T O$ such that $O^Z \leq_h O$.*

This contrasts with the fact that in the computability setting already a weakly 2-random set forms a minimal pair with $\emptyset'$.

4.4 Classifying Π^1_1-randomness within Δ^1_1-randomness

For each Π^1_1 class $\mathcal{S}$ we have $\mathcal{S} \subseteq \{Y\colon\, \omega_1^Y > \omega_1^{\mathrm{ck}}\} \cup \bigcup_{\alpha<\omega_1^{\mathrm{ck}}} \mathcal{S}_\alpha$, because $Y \in \mathcal{S}$ implies $Y \in \mathcal{S}_\alpha$ for some $\alpha < \omega_1^Y$. For the largest null Π^1_1 class $\mathcal{Q}$, equality holds because $\{Y\colon\, \omega_1^Y > \omega_1^{\mathrm{ck}}\}$ is a null Π^1_1 class:
$\mathcal{Q} = \{Y\colon\, \omega_1^Y > \omega_1^{\mathrm{ck}}\} \cup \bigcup_{\alpha<\omega_1^{\mathrm{ck}}} \mathcal{Q}_\alpha$.
For $\alpha < \omega_1^{\mathrm{ck}}$ the null class $\mathcal{Q}_\alpha$ is Δ^1_1 by the Approximation Lemma 2.3(ii). So, by de Morgan's, the foregoing fact yields a characterization of the Π^1_1-random sets within the Δ^1_1-random sets by a lowness property in the new setting.

Theorem 4.7. *Z is Π^1_1-random $\Leftrightarrow \omega_1^Z = \omega_1^{\mathrm{ck}}$ & Z is Δ^1_1-random.*

5 Lowness properties in higher computability theory

We study some properties that are closed downward under $\leq_h$, and relate them to higher randomness notions. The results are due to Chong, Nies and Yu (2008).

5.1 Hyp-dominated sets

Definition 5.1. *We say that A is hyp-dominated if each function $f \leq_h A$ is dominated by a hyperarithmetical function.*

A is hyp-dominated $\Rightarrow \omega_1^A = \omega_1^{\mathrm{ck}}$.
Weakly Δ^1_1 random means in no *closed* null Δ^1_1 class.

Theorem 5.2 (Kjos-Hanssen, Nies, Stephan, Yu (2009))**.** *Z is Π^1_1-random $\Leftrightarrow$ Z is hyp-dominated and weakly Δ^1_1-random.*

"$\Rightarrow$" is in the Book 9.4.3. "$\Leftarrow$" is a franco-stephanian domination argument using the Bounding Principle (see Book, Ex 9.4.6. and solution).

5.2 Traceability

The higher analogs of c.e., and of computable traceability coincide, again because of the Bounding Principle.

Definition 5.3. *(i) Let h be a non-decreasing Δ^1_1 function. A Δ^1_1* trace *with* bound h *is a uniformly Δ^1_1 sequence of sets $(T_n)_{n\in\omega}$ such that $\forall n\, \#T_n \leq h(n)$. $(T_n)_{n\in\omega}$ is a trace for the function f if $f(n) \in T_n$ for each n.*
(ii) A is Δ^1_1 traceable *if there is an unbounded non-decreasing hyperarithmetical function h such that each function $f \leq_h A$ has a Δ^1_1 trace with bound h.*

As usual, the particular choice of the bound h does not matter.

5.3 Examples

- Chong, Nies and Yu showed that there are $2^{\aleph_0}$ many Δ^1_1 traceable sets.
- In fact, each generic set for forcing with perfect Δ^1_1 trees (introduced in Sacks 4.5.IV) is Δ^1_1 traceable.
- Also, by Sacks 4.10.IV, there a generic set $Z \leq_h O$. Then Z is Δ^1_1 traceable and $Z \notin \Delta^1_1$.

5.4 Low(Δ^1_1-random)

Δ^1_1 traceability characterizes lowness for Δ^1_1-randomness. The following is similar to results of Terwijn/Zambella (1998) Kjos-Hanssen/Nies/Stephan (2007).

Theorem 5.4. *The following are equivalent for a set A.*

(i) A is Δ^1_1-traceable (or equivalently, Π^1_1 traceable).

(ii) Each null $\Delta^1_1(A)$ class is contained in a null Δ^1_1 class.

(iii) A is low for Δ^1_1-randomness.

(iv) Each Π^1_1-ML-random set is $\Delta^1_1(A)$-random.

5.5 Low(Π^1_1-random)

For each set A there is a largest null $\Pi^1_1(A)$ class $\mathcal{Q}(A)$ by relativizing Theorem 4.5. Clearly $\mathcal{Q} \subseteq \mathcal{Q}(A)$;
A is called *low for* Π^1_1*-randomness* iff they are equal.

Lemma 5.5. *If A is low for Π^1_1-randomness then $\omega_1^A = \omega_1^{\mathrm{ck}}$.*

Proof. Otherwise, $A \geq_h O$ by Theorem 2.8. By Corollary 4.6 there is a Π^1_1-random set $Z \leq_h O$, and Z is not even $\Delta^1_1(A)$ random.

Question 5.6. *Does lowness for Π^1_1-randomness imply being in Δ^1_1?*

5.6 Π^1_1-random cuppable

By the following result, lowness for Π^1_1-randomness *implies* lowness for Δ^1_1-randomness. We say that A is Π^1_1-random cuppable if $A \oplus Y \geq_h O$ for some Π^1_1-random set Y.

Theorem 5.7. *A is low for Π^1_1-randomness $\Leftrightarrow$*
(a) A is not Π^1_1-random cuppable & (b) A is low for Δ^1_1-randomness.

Proof.
$\Rightarrow$: *(a)* By Lemma 5.5 $A \not\geq_h O$. Therefore the $\Pi^1_1(A)$ class

$$\{Y : \ Y \oplus A \geq_h O\}$$

is null, by relativizing Cor. 2.9 to A. Thus A is not Π^1_1-random cuppable.
(b) Suppose for a contradiction that Y is Δ^1_1-random but $Y \in \mathcal{C}$ for a null $\Delta^1_1(A)$ class $\mathcal{C}$. The union $\mathcal{D}$ of all null Δ^1_1 classes is Π^1_1 by Martin-Löf (1970) (see Book Ex. 9.3.11). Thus Y is in the $\Sigma^1_1(A)$ class $\mathcal{C} - \mathcal{D}$.
By the Gandy Basis Theorem 2.10 relative to A there is $Z \in \mathcal{C} - \mathcal{D}$ such that $\omega_1^{Z \oplus A} = \omega_1^A = \omega_1^{\mathrm{ck}}$. Then Z is Δ^1_1-random but not $\Delta^1_1(A)$-random, so by Theorem 4.7 and its relativization to A, Z is Π^1_1-random but not $\Pi^1_1(A)$-random, a contradiction.
$\Leftarrow$: By Fact 4.4 relative to A we have

$$\mathcal{Q}(A) = \{Y : \ \omega_1^{Y \oplus A} > \omega_1^A\} \cup \textstyle\bigcup_{\alpha < \omega_1^A} \mathcal{Q}(A)_\alpha.$$

By hypothesis (a) $O \not\leq_h A$ and hence $\omega_1^A = \omega_1^{\mathrm{ck}}$, so

$$\omega_1^{Y \oplus A} > \omega_1^A \textit{ is equivalent to } O \leq_h A \oplus Y.$$

If Y is Π^1_1-random then firstly $O \not\leq_h A \oplus Y$ by (a), and secondly $Y \notin \mathcal{Q}(A)_\alpha$ for every $\alpha < \omega_1^A$ by hypothesis (b). Therefore $Y \notin \mathcal{Q}(A)$ and Y is $\Pi^1_1(A)$-random.