888 Algebraic Foundations of Data (Topics)

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A preview of the learning outcomes

Many optimization problems in data and learning are built on algebraic ideas. For example, principal component analysis finds a low-rank approximation of a matrix, a problem central to linear algebra. This course builds out from this example to study the algebraic foundations of several optimization problems to find representations and understanding of data.

Topics[1]

There are many topics listed here, some of which entire courses can be devoted to. Our goal in this course is to identify important problems for motivation and to highlight algebraic foundations of data. Lectures will be important to achieving this goal. Textbook supplementary references are provided for those who want to build up their background, and research articles are listed to point to areas that could be exciting to dive into. There are six required homeworks to cement understanding by engaging with the material at the start of the semester. As the topics become more specialized optional challenge problems will be posted to inspire new ideas and approaches.

Part one of the course is a linear algebra review. The second part of course uses SVD as a spring board into several different topics including Procrustes problems. From Topic 13 and onward, a selection of these topics will be made based on time permitting. In a perfect world, we would cover all of these, but the more likely scenario is to cover half.

This list is continuously updated to match the pace of the class:

- Topic 1: Least squares
  Supplementary References: VMLS–Chapter 12
  Outcomes: Derive several expressions for the solution of the least squares problem; review QR factorization; Pseudo-inverses and how to compute them from a QR factorization
  Lecture: 1

[1]Topics are subject to change.

- Topic 2: Least squares data fitting
  Supplementary References: VMLS-Chapter 13.1-13.2
  Outcomes: Find a mathematical model, or an approximate model, of some relation, given some observed data; Fitting univariate functions; polynomial fit; regression; validation and overfitting; AR model example (page 259, 266-268); Vandermonde matrices.
  Lecture: 2, 3

- Topic 3: Eigenvalues of symmetric matrices and Rayleigh–Ritz Quotients
  Supplementary References: ALA–Chapter 8.5 Chebfun resource; Guangliang Chen's Notes
  Outcomes: Quadratic forms, optimizing a quadratic form over the unit ball, power iteration, useful properties of derivaties in linear algebra; power iteration (ALAChapter 9.5), generalized Raleigh-Ritz Quotients
  Lecture: 3, 4

- Topic 4: Singular value decomposition
  Supplementary References: ALA–Chapter 8.7; NLA–Chapter 1 (Lecture 4); Stanford notes
  Outcomes: Data compression
  Lecture: 4, 5

- Topic 5: Principle component analysis
  Supplementary References: ALA–Chapter 8.8; EDRML–Chapter 5.1-5.2
  Outcomes: Low rank approximation
  Lecture: 6, 7

- Topic 6: Matrix groups and manifolds
  Supplementary References: Krishnamurthy notes; MGU–Chapter 3-7 (Orthogonal group; topology of matrix groups; Lie Algebras; matrix exponentiation; manifolds)
  Outcomes: Review matrix groups; background knowledge to dive deeper into manifold learning
  Lecture: 8, 9, 10

- Topic 7: Procrustes Problems: Satellite rotations
  Supplementary References: APAM-PCV– Chapter 2
  Outcomes: Putting Procrustes problems into practice
  Lecture: 10

- Topic 8: Eigenvalue methods polynomial system solving: multiplication maps
  Supplementary References: UAG–Division Algorithm, Chapter 2.2 and 2.4 (Theorem 2.4.5)
  Outcomes: Multivariate polynomial division; cosets; Finiteness Theorem (a,b);

  Lecture: 11, 12

- Topic 9: First order optimization
  Supplementary References: EDRML–Chapter 4.1-4.4, Chapter 5.4
  Outcomes: Nearest point problems for smooth surfaces
  Lecture: 13, 14

- Topic 10: Algebraic optimization and the singular locus
  Supplementary References: TBD
  Outcomes: Tensor eigenvalue problems; Euclidean distance degrees; low rank matrices
  Lecture: 15

- Topic 11: Projection and elimination theory
  Supplementary References: UAG, INA–Chapter 4
  Outcomes: Use projection with PCA; explain what this means in nonlinear algebra
  Lecture: 16, 17

- Topic 12: Continuous maximization for unbalanced Procrustes problems
  Supplementary References: TBD
  Outcomes: Bridge between convex and nonconvex optimization; Stiefel manifolds
  Lecture: 17, 18

- Topic 13: A snapshot of algebra for computer vision (Time permitting)
  Supplementary References: [7]; Stanford course
  Outcomes: Multiview varieties and minimal problems
  Lecture: 19

- Topic 14: Elimination for phylogenetics
  Supplementary References: UAG and AS-Sullivant
  Outcomes: Division algorithm revisited; graphs and their projections
  Lecture: 20, 21

- Topic 15: EM Algorithm and maximum likelihood estimation
  Supplementary References: [8]; AS-Sullivant
  Outcomes: Using data
  Lecture: 22, 23

- Topic 16: Maximum likelihood degrees and data discriminants
  Supplementary References: TBD
  Outcomes: Computing data discriminants; connections to physics
  Lecture: 24

- Topic 17: Kernel Methods
  Supplementary References: EDRML– Chapter 3
  Outcomes: Kernel methods
  Lecture: 25, 26

- Topic 18: Kernel Principle component analysis
  Supplementary References: EDRML–Chapter 5.4
  Outcomes: Kernel methods
  Lecture: 27, 28

- Topic 19: Grassmannians and likelihood geometry of determinantal point process
  Supplementary References: [3, 5] [6, 9, 1]
  Outcomes: Projected DPP; apply likelihood geometry
  Lecture: 29, 30

- Topic 20: Expressiveness of Relu neural networks
  Supplementary References: INA–Chapter 7; [11]
  Outcomes: Expressive power of Relu neural in terms of tropical geometry
  Lecture: 31, 32

- Topic 21: Tropical maximum likelihood estimation
  Supplementary References: INA–Chapter 7, [2, 10]
  Outcomes: Extremal statistics through an algebra lens
  Lecture: 33, 34

- Topic 22: Feedforward neural networks viewed as tensors
  Supplementary References: [4] INA–Chapter 9; Online resource
  Outcomes: Identifiability of polynomial neural networks
  Lecture: 35, 36

- Topic 23: Orthogonally decomposable tensors
  Supplementary References: TBD
  Outcomes: Alternative ways to compress data
  Lecture: 37, 38

- Topic 24: Sums of squares and SDP relaxations (Time permitting)
  Supplementary References: INA— Chapter 12
  Outcomes: Expanding the algebraic optimization toolkit
  Lecture: 39, 40

- Topic 25: Log-linear models and exact tests (Time permitting)
  Supplementary References: TBD
  Outcomes: Quantifying uncertainty using algebraic methods
  Lecture: 41, 42

References

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