ON INITIAL SEGMENT COMPLEXITY AND DEGREES OF RANDOMNESS

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ABSTRACT. One approach to understanding the fine structure of initial segment complexity was introduced by Downey, Hirschfeldt and LaForte. They define $X \leq_K Y$ to mean that $(\forall n)\ K(X \upharpoonright n) \leq K(Y \upharpoonright n) + O(1)$. The equivalence classes under this relation are the *K-degrees*. We prove that if $X \oplus Y$ is 1-random, then X and Y have no upper bound in the K-degrees (hence, no join). We also prove that n-randomness is closed upward in the K-degrees. Our main tool is another structure intended to measure the *degree of randomness* of real numbers: the *vL*-degrees. Unlike the K-degrees, many basic properties of the *vL*-degrees are easy to prove. We show that $X \leq_K Y$ implies $X \leq_{vL} Y$, so some results can be transferred. The reverse implication is proved to fail. The same analysis is also done for $\leq_C$, the analogue of $\leq_K$ for plain Kolmogorov complexity.

Two other interesting results are included. First, we prove that for any $Z \in 2^\omega$, a 1-random real computable from a 1-Z-random real is automatically 1-Z-random. Second, we give a plain Kolmogorov complexity characterization of 1-randomness. This characterization is related to our proof that $X \leq_C Y$ implies $X \leq_{vL} Y$.

1. Introduction

This paper is part of an ongoing project to understand the initial segment complexity of random real numbers (by which we mean elements of 2^ω). Several authors have investigated oscillations in the complexity of initial segments of 1-random (i.e., Martin-Löf random) reals, with respect to either plain or prefix-free Kolmogorov complexity (denoted by C and K, respectively). These include Martin-Löf [18, 19], Chaitin [1, 3], Solovay [28] and van Lambalgen [29].

Our approach is different. While previous work focuses on *describing* the behavior of the initial segment complexity of a real number, we instead focus on *interpreting* that behavior. We argue that the initial segment complexity of $X \in 2^\omega$ carries useful information about X. An obvious example is Schnorr's theorem that $X \in 2^\omega$ is 1-random iff $(\forall n)\ K(X \upharpoonright n) \geq n - O(1)$. A more recent example is the fact that $X \in 2^\omega$ is 2-random iff $(\exists^\infty n)\ C(X \upharpoonright n) \geq n - O(1)$ (see Miller [20]; Nies, Stephan and Terwijn [23]). These results raise obvious questions: can 1-randomness

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be characterized in terms of initial segment C-complexity—a long elusive goal—or 2-randomness in terms of initial segment K-complexity? We will give positive answers to both questions.

Many of our results will be stated in terms of the *K-degrees*, which were introduced by Downey, Hirschfeldt and LaForte [6, 7]. Write $X \leq_K Y$ if Y has higher initial segment prefix-free complexity than X (up to a constant). Formally, $(\forall n)\ K(X \upharpoonright n) \leq K(Y \upharpoonright n) + O(1)$. The induced partial order is called the K-degrees. Define the *C-degrees* in the same way. This brings us to the second major theme of this paper: *degrees of randomness.* What does it means to say that one real number is more random than another? Based on the intuition that higher complexity implies more randomness, one might think that $X \leq_K Y$ (or $X \leq_C Y$) means that Y is more random than X. We will provide some evidence supporting this view.

Our effort to connect the properties of a real number to its initial segment complexity culminates in Corollary 7.5, which states that $X \oplus Z$ is 1-random iff

$$(\forall n)\ C(X \upharpoonright n) + K(Z \upharpoonright n) \geq 2n - O(1).$$

Thus, the initial segment C-complexity of $X \in 2^\omega$ gives a complete accounting of the reals $Z \in 2^\omega$ such that $X \oplus Z$ is 1-random. By symmetry, the same information is implicit in the initial segment K-complexity of X.

We will see that the corollary says more, but first we introduce the *vL-degrees.* These are a slight variant of the LR-degrees, which were introduced by Nies [22]; see Section 3 for details. Write $X \leq_{vL} Y$ if $(\forall Z \in 2^\omega)[X \oplus Z$ is 1-random $\implies$ $Y \oplus Z$ is 1-random$]$. The induced partial order is called the *van Lambalgen degrees* (or vL-degrees) because the definition was motivated by a theorem of van Lambalgen (Theorem 3.1). These degrees offer an alternative way to gauge randomness, one based on the global properties of reals, not on their finite initial segments. Corollary 7.5 shows that $X \leq_C Y$ implies $X \leq_{vL} Y$, and again by symmetry, that $X \leq_K Y$ implies $X \leq_{vL} Y$. Because many properties of the vL-degrees are easily proved, this new structure is a useful tool in studying the K-degrees and C-degrees. For example, we will show that if $X \leq_{vL} Y$ and X is n-random (i.e., 1-random relative to $\emptyset^{(n-1)}$), then Y is also n-random. By the above implications, every real with higher initial segment complexity than an n-random real must also be n-random. As promised, this supports the assertion that reals with higher K-degree (or C-degree) are more random.

The article is organized as follows. Section 2 covers the necessary concepts from Kolmogorov complexity and Martin-Löf randomness. The van Lambalgen degrees are introduced in Section 3 and several basic properties are proved. Section 4 is a digression from the main topics of the paper; in it we prove that any 1-random real computable from a 1-Z-random real is automatically 1-Z-random. This follows easily from van Lambalgen's theorem if $Z \in 2^\omega$ has 1-random Turing degree, but the general case requires more work. In Section 5, we prove that $X \leq_K Y$ implies $X \leq_{vL} Y$ and derive several results about the K-degrees. As was observed above, this result follows from Corollary 7.5. But it is also an immediate consequence of Theorem 5.1, which in turn is used in the proof of Corollary 7.5. Section 6 offers three results that contrast the K-degrees and the vL-degree. In particular, Proposition 6.2 shows that 1-random reals that differ by a computable permutation need not be K-equivalent (although they must be vL-equivalent), which demonstrates the essentially "local" nature of the K-degrees. The final section focuses on plain complexity. We prove that $X \in 2^\omega$ is 1-random iff $(\forall n)\ C(X \upharpoonright n) \geq n - K(n) - O(1)$. Alternately, X is 1-random iff for every computable $g\colon \omega \to \omega$ such that $\sum_{n\in\omega} 2^{-g(n)}$ is finite, $(\forall n)\ C(X \upharpoonright n) \geq n - g(n) - O(1)$. Finally, we prove Corollary 7.5 and derive several consequences for the C-degrees.

We finish this section with a brief discussion of how our results fit in with the previous work on the K-degrees of 1-random reals. It follows from work of Solovay [28] that Chaitin's halting probability Ω has a different K-degree than any arithmetically random real. Hence, there are at least two K-degrees. Yu, Ding and Downey [32] proved that $\mu\{X \in 2^\omega : X \leq_K Y\} = 0$, for any $Y \in 2^\omega$. From this, they conclude that there are uncountably many 1-random K-degrees (an explicit construction of an antichain of size $2^{\aleph_0}$ is give in [31]). An early goal of the present research was to calculate the measure of $\{Y \in 2^\omega : X \leq_K Y\}$. It must be noted that this measure depends on the choice of $X \in 2^\omega$. If X is computable, then it is K-below every real, hence $\mu\{Y \in 2^\omega : X \leq_K Y\} = 1$. On the other hand, the result of Yu, Ding and Downey implies that $\mu\{X \oplus Y \in 2^\omega : X \leq_K Y\} = 0$. Hence, $\mu\{Y \in 2^\omega : X \leq_K Y\} = 0$ for almost all $X \in 2^\omega$. Now by an easy complexity calculation, the measure is zero when X is (weakly) 2-random. In fact, it follows from Corollary 5.3 (ii) that 1-randomness is sufficient. It should be noted that this condition does not characterize 1-randomness; it is easy to construct a non-1-random real $X \in 2^\omega$ for which $\mu\{Y \in 2^\omega : X \leq_K Y\} = 0$.

Several results in this paper produce incomparable 1-random K-degrees, but none prove the existence of *comparable* 1-random K-degrees. That is done in a companion paper [21], where we prove that for any 1-random $Y \in 2^\omega$, there is a 1-random $X \in 2^\omega$ such that $X <_K Y$ (in fact, $\lim_{n\to\infty} K(Y \upharpoonright n) - K(X \upharpoonright n) = \infty$). Another problem that is not addressed in this paper is whether the C-degrees and the K-degrees differ for 1-random reals. They are known to be different in general [28, 7], but their relationship on the 1-random reals remains entirely open.

2. Preliminaries

We begin with a brief review of effective randomness and Kolmogorov complexity. A more complete introduction can be found in Li and Vitanyi [17] or the upcoming monograph of Downey and Hirschfeldt [5]. We assume that the reader is familiar with the basics of computability theory and measure theory. Soare [26] and Oxtoby [24] are good resources for these subjects.

Martin-Löf [18] introduced the most successful notion of effective randomness for real numbers. A *Martin-Löf* test is a uniform sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of Σ^0_1 classes such that $\mu(\mathcal{G}_n) \leq 2^{-n}$. A real $X \in 2^\omega$ is said to *pass* a Martin-Löf test $\{\mathcal{G}_n\}_{n\in\omega}$ if $X \notin \bigcap_{n\in\omega} \mathcal{G}_n$. We say that $X \in 2^\omega$ is 1-*random* (or *Martin-Löf random*) if it passes all Martin-Löf tests. This notion generalizes naturally; for any $n \in \omega$ and oracle $Z \in 2^\omega$, define *n-Z-randomness* by replacing the Σ^0_1 classes with $\Sigma^0_n[Z]$ classes in Martin-Löf's definition. The two parameters are related by the jump operator:

Theorem 2.1 (Kurtz [13])**.** *For $n \in \omega$ and $Z \in 2^\omega$, n-Z-randomness is equivalent to 1-$Z^{(n-1)}$-randomness.*

We write *n-random* for n-$\emptyset$-random, or equivalently, 1-$\emptyset^{(n-1)}$-random.

Kolmogorov [10] and Solomonoff [27] defined the complexity of a finite string to be the length of its shortest description. Formally, we use a partial computable function $M: 2^{<\omega} \to 2^{<\omega}$ to "decode" descriptions. Then the *Kolmogorov complexity* of $\sigma \in 2^{<\omega}$ with respect to M is $C_M(\sigma) = \min\{|\tau|: M(\tau) = \sigma\}$. There is an essentially optimal choice for the decoding function: a partial computable $V: 2^{<\omega} \to 2^{<\omega}$ with the property that if $M: 2^{<\omega} \to 2^{<\omega}$ is any other partial computable function, then $(\forall \sigma \in 2^{<\omega})\ C_V(\sigma) \leq C_M(\sigma) + O(1)$. We call V the *universal machine* and call $C(\sigma) = C_V(\sigma)$ the *plain (Kolmogorov) complexity* of $\sigma \in 2^{<\omega}$.

Levin [15] and Chaitin [2] introduced a modified form of Kolmogorov complexity that has natural connections to the Martin-Löf definition of randomness. For finite binary strings $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \prec \tau$ to mean that σ is a proper prefix of τ. Similarly, $\sigma \prec X$ means that σ is an initial segment of $X \in 2^\omega$. A set of strings $D \subseteq 2^{<\omega}$ is *prefix-free* if $(\forall \sigma, \tau \in D)\ \sigma \not\prec \tau$. A partial function $M: 2^{<\omega} \to 2^{<\omega}$ is *prefix-free* if its domain is a prefix-free set. If M is prefix-free, then we write K_M instead of C_M for the Kolmogorov complexity with respect to M. As before, there is a *universal prefix-free machine* $U: 2^{<\omega} \to 2^{<\omega}$ that is optimal for prefix-free partial computable functions, in the sense that $(\forall \sigma \in 2^{<\omega})\ K_U(\sigma) \leq K_M(\sigma) + O(1)$, for any such function M. We write $K(\sigma)$ for $K_U(\sigma)$ and call it the *prefix-free complexity* of $\sigma \in 2^{<\omega}$.

It is well known that the 1-random reals can be characterized in terms of the prefix-free complexity of their initial segments.

Theorem 2.2 (Schnorr)**.** *$X \in 2^\omega$ is 1-random iff $(\forall n)\ K(X \upharpoonright n) \geq n - O(1)$.*

Theorem 7.1 gives a similar characterization in terms of plain complexity.

We now review some of the combinatorics of prefix-free complexity. The fact that U has prefix-free domain implies that $\sum_{\sigma \in 2^{<\omega}} 2^{-K(\sigma)} \leq 1$; this is *Kraft's inequality.* It is clear that $(\forall \sigma \in 2^{<\omega})\ C(\sigma) \leq |\sigma| + O(1)$, but this would clearly violate Kraft's inequality were it true of prefix-free complexity. Instead, the natural upper bound on K is given by the following result.

Lemma 2.3 (Chaitin [2])**.**

(i) $(\forall \sigma \in 2^{<\omega})\ K(\sigma) \leq |\sigma| + K(|\sigma|) + O(1)$.
(ii) $(\forall n)(\forall k)\ |\{\sigma \in 2^n : K(\sigma) \leq n + K(n) - k\}| \leq 2^{n-k+O(1)}$.

Observe that K is applied to natural numbers as well as to binary strings. This is possible because we identify finite binary strings with natural numbers. In particular, $\sigma \in 2^{<\omega}$ represents $n \in \omega$ if the binary expansion of $n+1$ is 1σ. Note that strings of length n are identified with numbers between $2^n - 1$ and $2^{n+1} - 2$. Having fixed a natural effective bijection between $2^{<\omega}$ and ω, we may view K as a function of ω when it is convenient.

Information content measures provide an alternative approach to prefix-free complexity. These were introduced by Levin and Zvonkin [33] and studied further by Levin [16, 15]. They are implicit in Chaitin [2] and the name comes from his later paper [3]. A function $\widehat{K}: \omega \to \omega \cup \{\infty\}$ is an *information content measure* if

(i) $\sum_{n \in \omega} 2^{-\widehat{K}(n)}$ converges (where $2^{-\infty} = 0$).
(ii) $\{\langle n, k \rangle : \widehat{K}(n) \leq k\}$ is computable enumerable.

Not only is K an information content measure (when viewed as a function of ω), but it is minimal [15]: if $\widehat{K}$ is another information content measure, then $(\forall n)\ K(n) \leq \widehat{K}(n) + O(1)$.

We write C^Z and K^Z for the relativizations of plain and prefix-free complexity to an oracle $Z \in 2^\omega$. The results mentioned above remain true in their relativized forms. In particular, $X \in 2^\omega$ is 1-Z-random iff $(\forall n)\ K^Z(X \restriction n) \geq n - O(1)$. The following result relates K^Z to unrelativized prefix-free complexity when $Z \in 2^\omega$ is 1-random.

Ample Excess Lemma. *Let $Z \in 2^\omega$ be 1-random.*
(i) $\sum_{n\in\omega} 2^{n-K(Z\restriction n)} < \infty$.
(ii) $(\forall n)\ K^Z(n) \leq K(Z \restriction n) - n + O(1)$.

Proof. (i) Note that, for any $m \in \omega$,

$$\sum_{\sigma\in 2^m}\sum_{n\leq m} 2^{n-K(\sigma\restriction n)} = \sum_{\sigma\in 2^m}\sum_{\tau\prec\sigma} 2^{|\tau|-K(\tau)}$$
$$= \sum_{\tau\in 2^{\leq m}} 2^{m-|\tau|}2^{|\tau|-K(\tau)} = 2^m \sum_{\tau\in 2^{\leq m}} 2^{-K(\tau)} \leq 2^m,$$

where the last step is Kraft's inequality. Therefore, for any $p \in \omega$, there are at most $2^m/p$ strings $\sigma \in 2^m$ for which $\sum_{n\leq m} 2^{n-K(\sigma\restriction n)} > p$. This implies that $\mathcal{G}_p = \left\{X \in 2^\omega \colon \sum_{n\in\omega} 2^{n-K(X\restriction n)} > p\right\}$ has measure at most $1/p$. Thus $\{\mathcal{G}_{2^k}\}_{k\in\omega}$—clearly a uniform sequence of Σ^0_1 classes—is a Martin-Löf test. Therefore, $Z \notin \mathcal{G}_{2^k}$ for some $k \in \omega$, and so $\sum_{n\in\omega} 2^{n-K(Z\restriction n)} \leq 2^k$.

(ii) Define $\widehat{K}(n) = K(Z \restriction n) - n$. Note that $\{\langle n, k\rangle \colon \widehat{K}(n) \leq k\}$ is computably enumerable from Z. Hence by (i), $\widehat{K}$ is an information content measure relative to Z. The result now follows from the minimality of K^Z among such functions. □

3. The van Lambalgen degrees

When is a given 1-random real *more random* than another? The K-degrees attempt to answer this question using initial segment complexity. In this section, we propose a different approach—one based on the global behavior of real numbers, rather than their local structure. Our definition will be motivated by the following result.

Theorem 3.1 (van Lambalgen [30])**.** *If $X, Y \in 2^\omega$, then $X \oplus Y$ is 1-random iff X is 1-random and Y is 1-X-random.*

Nies [22] defined $X \geq_{LR} Y$ to mean

$$(\forall Z \in 2^\omega)[Z \text{ is 1-}X\text{-random} \implies Z \text{ is 1-}Y\text{-random}].$$

By Theorem 3.1, if X and Y are both 1-random, then $X \geq_{LR} Y$ iff $X \oplus Z$ is 1-random implies that $Y \oplus Z$ is 1-random, for all $Z \in 2^\omega$. Taking this partial characterization of $\geq_{LR}$ as a definition, we write $X \leq_{vL} Y$ iff

$$(\forall Z \in 2^\omega)[X \oplus Z \text{ is 1-random} \implies Y \oplus Z \text{ is 1-random}].$$

We call the equivalence classes induced by this relation the *van Lambalgen degrees.*

The *vL*-degrees differ from the *LR*-degrees in two relatively minor ways: the least *vL*-degree contains exactly the non-1-random reals (part (ii) of Theorem 3.4), and on 1-random reals $\leq_{vL}$ is equivalent to $\geq_{LR}$. Both changes make $\leq_{vL}$ more plausible as a measure of relative randomness. For the first, note that $X \equiv_T Y$ implies $X \equiv_{LR} Y$, so every *LR*-degree contains non-1-random reals. On the other hand, the *vL*-degree of a 1-random real contains only 1-random reals. But why the reversal of the ordering on 1-random reals? We will see in Corollaries 5.2 and 7.6 that both $\leq_K$ and $\leq_C$ imply $\leq_{vL}$. So the direction of the ordering is appropriate for our purposes and $X \leq_{vL} Y$ can reasonably be interpreted as saying that Y is more random than X.

The fact that both initial segment notions refine the vL-degrees allows us to transfer facts about the vL-degrees to these other structures, which is useful because many basic properties of the vL-degrees are easy to prove from known results. In addition to Theorem 3.1, we will use the following two facts.

Theorem 3.2 (Kučera [11])**.** *There is a* 1*-random real in every Turing degree* $\geq \mathbf{0}'$.

Theorem 3.3 (Kučera and Terwijn [12])**.** *For every* $X \in 2^\omega$*, there is a* $W \not\leq_T X$ *so that every* 1-X*-random real is* 1-$X \oplus W$*-random.*

Theorem 3.4 (Basic properties of the vL-degrees)**.**

(i) *If* $X \leq_{vL} Y$ *and* X *is* n*-random, then* Y *is* n*-random.*
(ii) *The least* vL*-degree is* $\mathbf{0}_{vL} = \{X \in 2^\omega : X$ *is not* 1*-random*$\}$.
(iii) *If* $X \oplus Y$ *is* 1*-random, then* X *and* Y *have no upper bound in the* vL*-degrees.*
(iv) *If* $Y \leq_T X$ *and* Y *is* 1*-random, then* $X \leq_{vL} Y$.[1]
(v) *There are* 1*-random reals* $X \equiv_{vL} Y$ *but* $X <_T Y$.

Proof. (i) Assume that $X \leq_{vL} Y$ and X is n-random. First consider $n = 1$. Select a 1-X-random real Z. Then $X \oplus Z$ is 1-random, so $Y \oplus Z$ is 1-random. Thus Y is 1-random.

Now take $n > 1$. By Theorem 3.2, there is a 1-random real $Z \equiv_T \emptyset^{(n-1)}$. Then X is n-random $\implies$ X is 1-Z-random $\implies$ $X \oplus Z$ is 1-random $\implies$ $Y \oplus Z$ is 1-random $\implies$ Y is 1-Z-random $\implies$ Y is n-random.

(ii) Clearly, if X is not 1-random then $X \leq_{vL} Y$ for any real Y. If X is 1-random, then $X \not\leq_{vL} \emptyset$, or else $\emptyset$ would be 1-random by part (i). Therefore, $\mathbf{0}_{vL}$ consists exactly of the non-1-random reals.

(iii) Let $X \oplus Y$ be 1-random and assume, for a contradiction, that $X, Y \leq_{vL} Z$. Because $X \leq_{vL} Z$ and $X \oplus Y$ is 1-random, $Z \oplus Y$ is 1-random too. Therefore, $Y \oplus Z$ is 1-random. But $Y \leq_{vL} Z$, so $Z \oplus Z$ must also be 1-random, which is a contradiction.

(iv) Assume that $Y \leq_T X$ and Y is 1-random. For any $Z \in 2^\omega$, $X \oplus Z$ is 1-random $\implies$ Z is 1-X-random $\implies$ Z is 1-Y-random $\implies$ $Y \oplus Z$ is 1-random. Therefore, $X \leq_{vL} Y$.

(v) Pick any 1-random real $X \geq_T \emptyset'$ (e.g., the halting probability Ω; see below). By Theorem 3.3, there is a $W \not\leq_T X$ such that every 1-X-random real is 1-$X \oplus W$-random. Take a 1-random real $Y \equiv_T X \oplus W$; this is guaranteed to exist by Theorem 3.2. So $X <_T Y$. Also, $Z \oplus X$ is 1-random $\iff$ Z is 1-X-random $\iff$ Z is 1-$X \oplus W$-random $\iff$ Z is 1-Y-random $\iff$ $Z \oplus Y$ is 1-random. Therefore, $X \equiv_{vL} Y$. □

Corollary 3.5 (More basic properties of the vL-degrees)**.**

(i) *There is no join in the* vL*-degrees.*
(ii) *If* $X \oplus Y$ *is* 1*-random, then* $X \oplus Y <_{vL} X, Y$ *and* $X \mid_{vL} Y$.
(iii) *There is no maximal* vL*-degree.*
(iv) *There is no minimal* 1*-random* vL*-degree.*
(v) *Every finite poset can be embedded into* $(2^\omega, \leq_{vL})$.

[1]Nies [22] observed that $Y \leq_T X$ implies $Y \leq_{LR} X$, from which part (iv) of Theorem 3.4 follows.

Proof. (i) Immediate from part (iii) of Theorem 3.4.

(ii) By part (iii) of Theorem 3.4, $X \mid_{vL} Y$. By part (iv) of the same theorem, $X \oplus Y \leq_{vL} X, Y$. Therefore, $X \oplus Y <_{vL} X, Y$.

(iii) If $X = X_1 \oplus X_2$ is 1-random, then $X <_{vL} X_1, X_2$ by part (ii). So the vL-degree of X is not maximal.

(iv) Assume that X is 1-random. Take any 1-X-random real Y; thus $X \oplus Y$ is 1-random. By part (ii), $\emptyset <_{vL} X \oplus Y <_{vL} X$. So, there is no minimal 1-random vL-degree.

(v) Suppose that $\mathbb{P} = (P, \leq)$ is a finite partial order; let $P = \{p_i\}_{i<n}$. Pick a 1-random real $X = \bigoplus_{i<n} X_i$. For any $k < n$, define $F(k) = \{i \colon p_k \leq p_i\}$ and let $Y_k = \bigoplus_{i \in F(k)} X_i$. Let $g \colon P \to 2^\omega$ be defined by $g(p_k) = Y_k$. It suffices to prove that $p_j \leq p_k \iff Y_j \leq_{vL} Y_k$. If $p_j \leq p_k$, then $F(k) \subseteq F(j)$ so $Y_j \leq_{vL} Y_k$, by part (i). If $p_j \not\leq p_k$, then $k \notin F(j)$ and so $Y_j \oplus X_k$ is 1-random. But $Y_k \oplus X_k$ is not 1-random since $k \in F(k)$. So $Y_j \not\leq_{vL} Y_k$. □

Note that part (v) of the corollary implies that the Σ^0_1 theory of $(2^\omega, \leq_{vL})$ is decidable, as in Lerman [14].

We finish the section by considering the vL-degrees of specific reals. Chaitin [2] proposed the *halting probability* Ω of the universal prefix-free machine U as a natural example of a 1-random real. Formally, let $\Omega = \sum_{U(\sigma)\downarrow} 2^{-|\sigma|}$. It is known that $\Omega \equiv_T \emptyset'$. The construction of Ω has a natural relativization. For any oracle $Z \in 2^\omega$, let U^Z be universal for prefix-free machines relative to Z. Let $\Omega^Z = \sum_{U^Z(\sigma)\downarrow} 2^{-|\sigma|}$. Relativizing the usual proofs, Ω^Z is 1-Z-random and computable from Z'. Using the results above, there are several simple observations we can make about the vL-degrees of Ω, the columns of Ω, and $\Omega^{\emptyset^{(n)}}$ for $n \in \omega$.

First, it is worth asking if the vL-degree of Ω^Z is independent of the choice of U^Z. It is not. The vL-degree of Ω is well defined, but this is not even true for $\Omega^{\emptyset'}$. It can be proved (using ideas in [8]) that if $Z \in 2^\omega$ has 1-random degree, then different choices of U^Z can give values of Ω^Z that are 1-random *relative to each other*. Therefore, by Theorem 3.4 (iii), different versions of Ω^Z have different vL-degrees. Even for a uniform choice of universal machines, there must be oracles $Z_0, Z_1 \in 2^\omega$ that differ only finitely much but for which Ω^{Z_1} and Ω^{Z_2} are relatively random [8], hence have no upper bound in the vL-degrees. Despite these issues, the results below depend only on the most elementary properties of Ω^Z and are independent of the choices of universal machines.

Corollary 3.6 (The vL-degrees of distinguished reals)**.**

(i) *Ω is the least Δ^0_2 1-random real in the vL-degrees.*
(ii) *For $n > 1$, an n-random and a Δ^0_n 1-random have no $\leq_{vL}$ upper bound.*
(iii) *If $m \neq n$, then $\Omega^{\emptyset^{(n)}}$ and $\Omega^{\emptyset^{(m)}}$ have no upper bound in the vL-degrees.*
(iv) *If $\Omega = \bigoplus_{n\in\omega} \Omega_n$, then $\{\Omega_n\}_{n\in\omega}$ is a vL-antichain of Δ^0_2 1-random reals (and again, no two have an upper bound in the vL-degrees).*

Proof. (i) follows from part (iv) of Theorem 3.4 and the fact that $\Omega \equiv_T \emptyset'$. For (ii), assume that $X \in 2^\omega$ is a Δ^0_n 1-random real. If $Y \in 2^\omega$ is n-random, then Y is 1-X-random. Now apply Theorem 3.4 (iii). In (iii), we can assume that $m > n$. Note that $\Omega^{\emptyset^{(n)}}$ is a Δ^0_{n+1} 1-random real and that $\Omega^{\emptyset^{(m)}}$ is $(n+1)$-random. So, (ii) implies (iii). Finally, (iv) follows from Theorem 3.4 (iii). □

4. Digression: the Turing degrees of 1-Z-random reals

The point of departure for this section is the observation that Theorem 3.4 has a simple but surprising consequence that is of interest independent from the study of degrees of randomness.

Corollary 4.1. *If X is n-random and $Y \leq_T X$ is 1-random, then Y is n-random.*

Proof. Immediate from parts (i) and (iv) of Theorem 3.4. $\square$

This result is relatively counterintuitive; it seems to say that it is possible to ensure a high degree of randomness by bounding the Turing degree of a 1-random real from *above*.

Corollary 4.1 has a parallel in the context of genericity: if X is n-generic and $Y \leq_T X$ is 2-generic, then Y is n-generic.[2] The fact that it is not sufficient to assume that Y is 1-generic is the subject of a recent paper of Csima, Downey, Greenberg, Hirschfeldt, and Miller [4].

Note that Nies, Stephan and Terwijn [23, Theorem 3.10] showed that a set A is 2-random iff A is 1-random and low for Ω (i.e., Ω is 1-A-random). From this result—a consequence of van Lambalgen's theorem—and the fact that the low for Ω sets are obviously closed downwards under Turing reducibility, Corollary 4.1 can be concluded for $n = 2$.

We wish to generalize the corollary from n-randomness to 1-Z-randomness for an arbitrary $Z \in 2^\omega$. It is easy to prove this generalization if Z has 1-random Turing degree; it follows from essentially the same argument that we used in the proof of Theorem 3.4. In particular, assume that X is 1-Z-random and that $Y \leq_T X$ is 1-random. Furthermore, assume (without loss of generality) that Z is 1-random. Then Z is 1-X-random, by van Lambalgen's theorem. So Z is 1-Y-random, which implies that Y is 1-Z-random, completing the argument.

A somewhat more complicated proof is necessary to remove the requirement that Z has 1-random degree.[3]

Lemma 4.2. *If $X \in 2^\omega$ is 1-random, then*

$$(\forall e)(\exists c)(\forall n)\ \mu\{A \in 2^\omega : \varphi_e^A \upharpoonright n = X \upharpoonright n\} \leq 2^{-n+c}.$$

Proof. Fix an index e. Uniformly define a family $\{\mathcal{H}_\sigma\}_{\sigma \in 2^{<\omega}}$ of Σ^0_1 classes by $\mathcal{H}_\sigma = \{A \in 2^\omega : \varphi_e^A \upharpoonright |\sigma| = \sigma\}$. Note that if σ and τ are incomparable strings, then

[2] We thank the referee for pointing out this result.

[3] The reader might hope to reduce Theorem 4.3 to the case solved above by conjecturing that if X is a 1-Z-random real, for some $Z \in 2^\omega$, then there is a 1-random real $\widehat{Z} \geq_T Z$ such that X is 1-$\widehat{Z}$-random. Although this would solve our problem, it is not true in general.

For a counterexample, take $X = \Omega$ and let Z be any non-computable Δ^0_2 *low for random* real; i.e., a real such that every 1-random real is 1-Z-random. These were first constructed in [12]. By definition, Ω is 1-Z-random. Now take any 1-random real $\widehat{Z} \in 2^\omega$ such that Ω is 1-$\widehat{Z}$-random; we will prove that $\widehat{Z} \not\geq_T Z$. By van Lambalgen's theorem, $\widehat{Z}$ is 1-Ω-random. But $\Omega \equiv_T \emptyset'$, so $\widehat{Z}$ is 2-random. Fix $e \in \omega$ and consider the class $\mathcal{G} = \{A \in 2^\omega : \varphi_e^A = Z\}$. Because Z is a Δ^0_2 set, it is the limit of a computable sequence $\{Z_s\}_{s \in \omega}$ of finite sets. Thus

$$\mathcal{G} = \{A \in 2^\omega : (\forall n)(\forall t)(\exists s \geq t)\ \varphi_{e,s}^A \upharpoonright n = Z_s \upharpoonright n\},$$

so $\mathcal{G}$ is a Π^0_2 class. A result of Sacks [25] states that $\mu\{A \in 2^\omega : A \geq_T Z\} = 0$ because Z is not computable. Hence, $\mu\mathcal{G} = 0$. Kurtz [13] observed that no 2-random real is contained in a measure zero Π^0_2 class, so $\varphi_e^{\widehat{Z}} \neq Z$. But the choice of e was arbitrary, proving that $\widehat{Z} \not\geq_T Z$.

$\mathcal{H}_\sigma \cap \mathcal{H}_\tau = \emptyset$. Now for each $i \in \omega$, define

$$F_i = \{\sigma \in 2^{<\omega} \colon \mu\mathcal{H}_\sigma > 2^{-|\sigma|+i}\}.$$

Note that the sets $F_i \subseteq 2^{<\omega}$ are uniformly computably enumerable and thus $\mathcal{G}_i = [F_i]$ is a uniform sequence of Σ^0_1 classes. We claim that $\mu\mathcal{G}_i \leq 2^{-i}$. Assume not. Then there is a prefix-free set $D \subseteq F_i$ such that $\mu[D] > 2^{-i}$. For distinct $\sigma, \tau \in D$, we have $\mathcal{H}_\sigma \cap \mathcal{H}_\tau = \emptyset$. Therefore,

$$\begin{aligned}\mu\{A \in 2^\omega \colon (\exists \sigma \in D)\ \varphi^A_e \restriction |\sigma| = \sigma\} &= \sum_{\sigma \in D} \mu\mathcal{H}_\sigma \\ &> \sum_{\sigma \in D} 2^{-|\sigma|+i} = 2^i \sum_{\sigma \in D} 2^{-|\sigma|} = 2^i \mu[D] > 2^i 2^{-i} = 1.\end{aligned}$$

This is a contradiction, so $\mu\mathcal{G}_i \leq 2^{-i}$. Therefore, $\{\mathcal{G}_i\}_{i\in\omega}$ is a Martin-Löf test. Now let $X \in 2^\omega$ be 1-random. Then $X \notin \mathcal{G}_c$ for some c. In other words,

$$(\forall n)\ \mu\{A \in 2^\omega \colon \varphi^A_e \restriction n = X \restriction n\} \leq 2^{-n+c},$$

which completes the proof. $\square$

Theorem 4.3. *For every $Z \in 2^\omega$, every 1-random real Turing reducible to a 1-Z-random real is also 1-Z-random.*

Proof. Take $X, Y \in 2^\omega$ such that X is 1-random and $X \leq_T Y$. Fix an index e such that $X = \varphi^Y_e$ and let c be the constant guaranteed by the previous lemma for this choice of X and e. Now uniformly enumerate, for every $\sigma \in 2^{<\omega}$, a set of strings $F_\sigma \subseteq 2^{<\omega}$ as follows. Search for strings $\sigma, \tau \in 2^{<\omega}$ such that $\varphi^\tau_e \restriction |\sigma| = \sigma$ and the use of $\varphi^\tau_e \restriction |\sigma|$ is exactly τ. Whenever such strings are found, put τ into F_σ *provided that this maintains the condition*

$$\sum_{\tau \in F_\sigma} 2^{-|\tau|} \leq 2^{-|\sigma|+c}. \tag{1}$$

Note that each F_σ is prefix-free, hence $\mu[F_\sigma] = \sum_{\tau \in F_\sigma} 2^{-|\tau|}$. Furthermore, it is clear that $[F_{X\restriction n}] = \{A \in 2^\omega \colon \varphi^A_e \restriction n = X \restriction n\}$, for every n. This is because, by our choice of c, condition (1) does not prevent the addition of any strings to $F_{X\restriction n}$.

Now consider $Z \in 2^\omega$ such that X is not 1-Z-random. Our goal is to prove that Y is also not 1-Z-random. As usual, let K^Z denote prefix-free Kolmogorov complexity relative to Z. For each $i \in \omega$, define a $\Sigma^0_1[Z]$ class

$$\mathcal{G}_i = \bigcup_{K^Z(\sigma) \leq |\sigma| - c - i} [F_\sigma].$$

Then

$$\mu\mathcal{G}_i \leq \sum_{K^Z(\sigma) \leq |\sigma| - c - i} \mu[F_\sigma] \leq \sum_{K^Z(\sigma) \leq |\sigma| - c - i} 2^{-|\sigma|+c} \leq \sum_{\sigma \in 2^{<\omega}} 2^{-K^Z(\sigma)-i} \leq 2^{-i}.$$

Therefore, $\{\mathcal{G}_i\}_{i\in\omega}$ is a Martin-Löf test relative to Z. Because X is not 1-Z-random, for each $i \in \omega$ there is a n such that $K^Z(X\restriction n) \leq n - c - i$. But then $Y \in \{A \in 2^\omega \colon \varphi^A_e \restriction n = X \restriction n\} = [F_{X\restriction n}] \subseteq \mathcal{G}_i$. This is true for all i, so Y is not 1-Z-random. $\square$

5. Prefix-free complexity and the K-degrees

What properties of $X \in 2^\omega$ are implicit in the prefix-free complexity of its initial segments? We begin this section with a partial answer to this question; we show that the initial segment complexity of X determines, for any $Z \in 2^\omega$, whether $X \oplus Z$ is 1-random. This proves that $X \leq_K Y$ implies $X \leq_{vL} Y$, so the results of the previous sections have consequences in the K-degrees. For example, Corollary 5.3 (i) implies that the prefix-free complexity of the initial segments of X determine whether of not X is n-random.

For the statement of the main theorem, recall that we use strings of length n to represent the numbers between $2^n - 1$ and $2^{n+1} - 2$. We also need some additional notation for the proof of the theorem. Define $X \widehat{\oplus} Z$ to be

$$\langle z_0, x_0, z_1, x_1, x_2, z_2, x_3, x_4, x_5, x_6, z_3, \ldots\ldots, z_n, x_{2^n-1}, \ldots, x_{2^{n+1}-2}, z_{n+1}, \ldots\ldots\rangle,$$

where $X = \langle x_0, x_1, x_2, \ldots\rangle$ and $Z = \langle z_0, z_1, z_2, \ldots\rangle$. It is easy to see that the class of 1-random reals is closed under computable permutations of ω. Therefore, $X \widehat{\oplus} Z$ is 1-random iff $X \oplus Z$ is 1-random. In fact, $X \widehat{\oplus} Z \equiv_{vL} X \oplus Z$. We can also define $\sigma \widehat{\oplus} \tau$ for strings $\sigma, \tau \in 2^{<\omega}$, provided that $2^{|\tau|-1} - 1 \leq |\sigma| \leq 2^{|\tau|} - 1$.

Theorem 5.1. *$X \oplus Z$ is 1-random iff $(\forall n)\ K(X \upharpoonright (Z \upharpoonright n)) \geq Z \upharpoonright n + n - O(1)$.*

Proof. First, assume that $X \oplus Z$ is 1-random. Then $X \widehat{\oplus} Z$ is also 1-random. Note that $K(X \upharpoonright (Z \upharpoonright n)) = K((X \upharpoonright (Z \upharpoonright n)) \widehat{\oplus} (Z \upharpoonright n)) + O(1)$ (the definition of $\widehat{\oplus}$ is contrived to ensure that $(X \upharpoonright (Z \upharpoonright n)) \widehat{\oplus} (Z \upharpoonright n$ is well defined). But $(X \upharpoonright (Z \upharpoonright n)) \widehat{\oplus} (Z \upharpoonright n) = (X \widehat{\oplus} Z) \upharpoonright (Z \upharpoonright n + n)$, so $K(X \upharpoonright (Z \upharpoonright n)) = K((X \widehat{\oplus} Z) \upharpoonright (Z \upharpoonright n + n)) + O(1) \geq Z \upharpoonright n + n - O(1)$, for all n.

For the other direction, define a prefix-free machine $M \colon 2^{<\omega} \to 2^{<\omega}$ as follows. To compute $M(\tau)$, look for τ_1, τ_2, η_1 and η_2 such that $\tau = \tau_1\tau_2$, $U(\tau_1) = \eta_1 \widehat{\oplus} \eta_2$ and $|\eta_1\tau_2| = \eta_2$. If these are found, define $M(\tau) = \eta_1\tau_2$.

Assume that $X \oplus Z$ is not 1-random. Then for each k, there is an m such that $K((X \widehat{\oplus} Z) \upharpoonright m) \leq m - k$. Take strings η_1 and η_2 such that $\eta_1 \widehat{\oplus} \eta_2 = (X \widehat{\oplus} Z) \upharpoonright m$ and let τ_1 be a minimal U-program for $\eta_1 \widehat{\oplus} \eta_2$. Let $n = |\eta_2|$. Note that $|\eta_1| \leq 2^n - 1$ and that $\eta_2 \geq 2^n - 1$. So, there is a string τ_2 such that $\eta_1\tau_2 = X \upharpoonright \eta_2$. Then $M(\tau_1\tau_2) = X \upharpoonright \eta_2$. Therefore,

$$\begin{aligned}
K(X \upharpoonright (Z \upharpoonright n)) &\leq K(X \upharpoonright \eta_2) \leq K_M(X \upharpoonright \eta_2) + O(1) \leq |\tau_1\tau_2| + O(1) \\
&\leq K(\eta_1 \widehat{\oplus} \eta_2) + |\tau_2| + O(1) \leq |\eta_1\eta_2| - k + |\tau_2| + O(1) \\
&= |\eta_1\tau_2| + |\eta_2| - k + O(1) = \eta_2 + |\eta_2| - k + O(1) = Z \upharpoonright n + n - k + O(1),
\end{aligned}$$

where the constant depends only on M. Because k was arbitrary, $K(X \upharpoonright (Z \upharpoonright n)) - Z \upharpoonright n - n$ is not bounded below. Therefore, $(\forall n)\ K(X \upharpoonright (Z \upharpoonright n)) \geq Z \upharpoonright n + n - O(1)$ implies that $X \oplus Z$ is 1-random. $\square$

The following corollary is immediate.

Corollary 5.2. *$X \leq_K Y \implies X \leq_{vL} Y$.*

We see in the next section that the reverse implication fails. Combined with Theorem 3.4 and Corollaries 3.5 and 3.6, this corollary has interesting implications in the K-degrees.

Corollary 5.3.

(i) *If $X \leq_K Y$ and X is n-random, then Y is n-random.*
(ii) *If $X \oplus Y$ is* 1*-random, then $X \mid_K Y$ and X and Y have no upper bound in the K-degrees. Therefore, there is no join in the K-degrees.*
(iii) *If $m \neq n$, then $\Omega^{\emptyset^{(n)}}$ and $\Omega^{\emptyset^{(m)}}$ have no upper bound in the K-degrees.*
(iv) *If $\Omega = \bigoplus_{n\in\omega} \Omega_n$, then $\{\Omega_n\}_{n\in\omega}$ is a K-antichain of Δ^0_2 1-random reals (and again, no two have an upper bound in the K-degrees).*

R. Rettinger has independently announced the first part of (ii): that if $X \oplus Y$ is 1-random, then $X \mid_K Y$. Part (iii) of the corollary extends a result of Yu, Ding and Downey (with Denis Hirschfeldt) [32]; they proved that if $p < q$, then $\Omega^{\emptyset^{(q)}} \not\leq_K \Omega^{\emptyset^{(p)}}$. Other connections to [32] were discussed in the introduction.

Part (i) of Corollary 5.3 implies that n-randomness has a characterization in terms of initial segment K-complexity, although not necessarily an elegant one. As an example, we give an explicit characterization of 2-randomness. By van Lambalgen's theorem $X \oplus \Omega$ is 1-random iff X is 1-Ω-random. But $\Omega \equiv_T \emptyset'$, so $X \oplus \Omega$ is 1-random iff X is 2-random. Hence, by Theorem 5.1:

Corollary 5.4. *X is* 2*-random iff* $(\forall n)\ K(X \restriction (\Omega \restriction n)) \geq \Omega \restriction n + n - O(1)$.

6. Contrasting the K-degrees and vL-degrees

The connection between the K-degrees and the vL-degrees has proved useful in understanding the K-degrees, but it provides only part of the picture. In this section, we prove three results that contrast $\leq_K$ and $\leq_{vL}$. One consequence will be that $\leq_{vL}$ does not, in general, imply $\leq_K$, even for Δ^0_2 1-random reals. By Corollary 3.6, Ω is vL-below every other Δ^0_2 1-random real. On the other hand, our first result implies that $\Omega \not\leq_K \Omega_0$, where $\Omega = \Omega_0 \oplus \Omega_1$. Furthermore, Proposition 6.2 gives us a Δ^0_2 1-random real $X \in 2^\omega$ such that X and Ω have no upper bound in the K-degrees.

Once again, recall that the string $\sigma \in 2^{<\omega}$ represents the natural number $1\sigma - 1$. So, strings of length n represent numbers between $2^n - 1$ and $2^{n+1} - 2$, which implies that $\sigma \geq |\sigma|$ for every $\sigma \in 2^{<\omega}$.

Proposition 6.1. *If $X \oplus Y$ is* 1*-random, then $X \mid_K X \oplus Y$.*

Proof. Assume that $X \oplus Y$ is 1-random. It follows from Corollary 5.2 and part (ii) of Corollary 3.5 that $X \not\leq_K X \oplus Y$.

Now assume, for a contradiction, that $X \oplus Y <_K X$. Fix n and consider $\sigma = \sigma_1\sigma_2\sigma_3 \prec X$ such that $|\sigma_1| = 2^{n+1}$, $|\sigma_2| = n$ and $|\sigma_2\sigma_3| = \sigma_2$. The last condition is consistent because $\sigma_2 \geq |\sigma_2|$. The idea of the proof is to show that σ has a short description, and hence $X \oplus Y \restriction |\sigma|$ does too. But we can compute σ_2 from $|\sigma|$. Just take the unique number n such that $2^{n+1} < |\sigma| < 2^{n+2}$; then $\sigma_2 = |\sigma| - 2^{n+1}$. Thus $|\sigma|$ encodes n bits of X that are not in $X \oplus Y \restriction |\sigma|$, from which we refute the randomness of $X \oplus Y$.

We turn to the details. First note that we can determine both n and $|\sigma|$ from σ_2, so $K(\sigma) \leq K(\sigma_2) + |\sigma_1\sigma_3| + O(1) \leq |\sigma_2| + K(n) + |\sigma_1\sigma_3| + O(1) \leq |\sigma| + 2\log n + O(1)$, where the constant does not depend on n. Therefore, $K(X \oplus Y \restriction |\sigma|) \leq |\sigma| + 2\log n + O(1)$. Next we compute $K(X \oplus Y \restriction 2|\sigma|)$. Note that $|\sigma| < 2^{n+2}$, so $X \oplus Y \restriction |\sigma|$ contains at most the first 2^{n+1} bits of X, hence no part of σ_2. On the other hand, $X \oplus Y \restriction 2|\sigma|$ contains all of σ_2. From $X \oplus Y \restriction |\sigma|$ we can compute $|\sigma|$, and hence σ_2. Therefore, to compute $X \oplus Y \restriction 2|\sigma|$ from $X \oplus Y \restriction |\sigma|$ we need to store only $|\sigma| - |\sigma_2|$

unknown bits. Thus $K(X \oplus Y \upharpoonright 2|\sigma|) \leq K(X \oplus Y \upharpoonright |\sigma|) + |\sigma| - |\sigma_2| + O(1) \leq |\sigma| + 2\log n + |\sigma| - n + O(1) = 2|\sigma| + 2\log n - n + O(1)$. Again, the constant does not depend on n. Because n is arbitrary, $X \oplus Y$ is not 1-random. This contradicts our hypothesis, proving that $X \oplus Y \not\leq_K X$. Therefore, $X \mid_K X \oplus Y$. □

In light of the other results in this section, one might hope to improve Proposition 6.1 to show that X and $X \oplus Y$ have no upper bound in the K-degrees. Surprisingly, this need not be the case. In the companion paper [21], it is proved that for every 1-random $Z \in 2^\omega$, there is another 1-random $X \in 2^\omega$ such that $X <_K Z$ and for every $Y \in 2^\omega$ we also have $X \oplus Y <_K Z$. By taking Y to be 1-X-random, we get an example where $X \oplus Y$ is 1-random (by van Lambalgen's theorem) but Z bounds both X and $X \oplus Y$ in the K-degrees.

We turn to the second result of the section. A computable permutation of a 1-random real is also 1-random; hence by Theorem 3.4 (iv), vL-degrees are invariant under computable permutations of ω. On the other hand, the next proposition shows that 1-random reals $X, Y \in 2^\omega$ that differ only by a computable permutation can have K-degrees with no upper bound. In fact, we prove that there is a fixed permutation f of the natural numbers such that X and $f(X)$ have no K-upper bound for every 1-random $X \in 2^\omega$. This calls into question the validity of the K-degrees as a measure of the *degree of randomness* of random reals. If one believes that computably isomorphic 1-random reals should be equivalently random, then the K-degrees are too strong.

To understand the proof of Proposition 6.2, one should think of it as a modification of a direct proof that could have been given for Corollary 5.3 (ii): if X and Y are 1-random relative to each other, then they have no upper bound in the K-degrees. Of course, if $f\colon \omega \to \omega$ is a computable permutation, then X and $f(X)$ are definitely not mutually 1-random, so it may not be clear how the corollary relates to the current situation. The idea is to define an f that switches pairs of disjoint blocks of ω such that each pair of blocks is large enough to make the smaller blocks insignificant in comparison. Then for any 1-random real $X \in 2^\omega$, there are long initial segments of X and $f(X)$ that behave sufficiently as if they were mutually 1-random (because the vast majority of the bits are from disjoint parts of X). These initial segments are enough to prove that no real can be K-above both X and $f(X)$.

Proposition 6.2. *There is a computable permutation $f\colon \omega \to \omega$ such that if $X \in 2^\omega$ is 1-random, then X and $f(X)$ have no upper bound in the K-degrees.*

Proof. First define a computable sequence $\{a_n\}_{n\in\omega}$ by recursion; let $a_0 = 0$ and $a_{n+1} = a_n + 2^{2a_n+2}$, for all $n \in \omega$. Define a computable permutation of ω by

$$f(m) = \begin{cases} m + 2^{2a_n+1}, & \text{if } n \in \omega \text{ and } a_n \leq m < a_n + 2^{2a_n+1} \\ m - 2^{2a_n+1}, & \text{if } n \in \omega \text{ and } a_n + 2^{2a_n+1} \leq m < a_{n+1}. \end{cases}$$

Now let $X \in 2^\omega$ and assume that $Z \in 2^\omega$ is K-above both X and $f(X)$. We will prove that X is not 1-random. Fix $n \in \omega$. Then $K(X \upharpoonright (a_n + Z \upharpoonright 2a_n)) \leq K(Z \upharpoonright (a_n + Z \upharpoonright 2a_n)) + O(1) \leq a_n + Z \upharpoonright 2a_n + K(n) + O(1)$. For the same reason, $K(f(X) \upharpoonright (a_n + Z \upharpoonright 2a_n)) \leq a_n + Z \upharpoonright 2a_n + K(n) + c$, for some $c \in \omega$ that does not depend on n. By Lemma 2.3 (ii), the number of strings $\tau \in 2^{a_n + Z \upharpoonright 2a_n}$ such that

$K(\tau) \le a_n + Z\upharpoonright 2a_n + K(n) + c$ is bounded by $2^{a_n + Z\upharpoonright 2a_n - k + O(1)}$, where

$$\begin{aligned} k &= (a_n + Z\upharpoonright 2a_n + K(a_n + Z\upharpoonright 2a_n)) - (a_n + Z\upharpoonright 2a_n + K(n) + c) \\ &= K(Z\upharpoonright 2a_n) - K(n) - O(1) \ge 2a_n - K(n) - O(1). \end{aligned}$$

Therefore, the number of such τ is bounded by $2^{Z\upharpoonright 2a_n + K(n) - a_n + d}$, for some $d \in \omega$ that is again independent of n. We will now show that

$$K(X\upharpoonright(a_n + 2^{2a_n+1} + Z\upharpoonright 2a_n)) \le a_n + 2^{2a_n+1} + Z\upharpoonright 2a_n - (a_n - 2K(n)) + O(1),$$

where the constant is independent of n. Because $\lim_{n\to\infty} a_n - 2K(n) = \infty$, this proves that X is not 1-random.

We may assume that the universal machine U was chosen so that for every $\sigma \in 2^{<\omega}$, there are U-programs for σ of every length greater that $K(\sigma)$. (It suffices to define U by $U(0^i 1\sigma) = \widehat{U}(\sigma)$, for every $i \in \omega$ and $\sigma \in 2^{<\omega}$, where $\widehat{U}$ is an arbitrary universal machine.) So there is a U-program $\sigma_1 \in 2^{a_n + Z\upharpoonright 2a_n + K(n) + O(1)}$ for $X\upharpoonright(a_n + Z\upharpoonright 2a_n)$, from which we can also determine n, a_n, $K(n)$ and $Z\upharpoonright 2a_n$. Now we can effectively enumerate the strings of length $a_n + Z\upharpoonright 2a_n$ with prefix-free complexity bounded by $a_n + Z\upharpoonright 2a_n + K(n) + c$. Let $\sigma_2 \in 2^{Z\upharpoonright 2a_n + K(n) - a_n + d}$ code the position of $f(X)\upharpoonright(a_n + Z\upharpoonright 2a_n)$ in this list. Given σ_1 and σ_2, since $Z\upharpoonright 2a_n \le 2^{2a_n+1}$, we can reconstruct $a_n + 2\cdot Z\upharpoonright 2a_n$ bits of $X\upharpoonright(a_n + 2^{2a_n+1} + Z\upharpoonright 2a_n)$; take σ_3 to be the remaining bits. Finally, note that σ_1 is self-delimiting and that from σ_1 we can compute the lengths of σ_2 and σ_3. Therefore,

$$\begin{aligned} &K(X\upharpoonright(a_n + 2^{2a_n+1} + Z\upharpoonright 2a_n)) \le |\sigma_1\sigma_2\sigma_3| + O(1) \\ &= (a_n + Z\upharpoonright 2a_n + K(n)) + (Z\upharpoonright 2a_n + K(n) - a_n) + (2^{2a_n+1} - Z\upharpoonright 2a_n) + O(1) \\ &\qquad = a_n + 2^{2a_n+1} + Z\upharpoonright 2a_n - (a_n - 2K(n)) + O(1). \end{aligned}$$

This completes the proof. □

The final result of this section is less elegant than the previous results, but it is also more general.

Proposition 6.3. *For any finite collection $X_0, \dots, X_k$ of 1-random reals, there is another 1-random real $Y \le_T X_0 \oplus \dots \oplus X_k \oplus \emptyset'$ such that, for every $i \le k$, Y and X_i have no upper bound in the K-degrees.*

Proof. Let $\mathcal{R} = \{Z \in 2^\omega : (\forall n)\ K(Z\upharpoonright n) \ge n\}$ and note that $\mu\mathcal{R} \ge 1/2$. We define two predicates:

$$A(\tau, p) \iff \mu\{Z \succ \tau : Z \notin \mathcal{R}\} > p$$

$$\text{and } B(\sigma, s) \iff (\forall i \le k)(\forall \tau \in 2^{|\sigma|}) \left[\begin{array}{c} (\exists n < |\sigma|)\ K(\sigma\upharpoonright n) > K(\tau\upharpoonright n) + s \\ \vee\ (\exists n < |\sigma|)\ K(X_i\upharpoonright n) > K(\tau\upharpoonright n) + s \end{array} \right],$$

where $\sigma, \tau \in 2^{<\omega}$, $p \in \mathbb{Q}$ and $s \in \omega$. Note that if Z and X_i have no upper bound in the K-degrees, for every $i \le k$, then by compactness, there is an n such that $B(Z\upharpoonright n, s)$. It should be clear that $B(\sigma, s)$ is uniformly decidable from $X_0 \oplus \dots \oplus X_k \oplus \emptyset'$. To see that $A(\tau, p)$ can be decided by $\emptyset'$, note that it is equivalent to $(\exists s)\ \mu\{Z \succ \tau : (\exists n \le s)\ K_s(Z\upharpoonright n) < n\} > p$. We construct $Y = \bigcup_{s\in\omega} \sigma_s$ by finite initial segments $\sigma_s \in 2^{<\omega}$ such that $B(\sigma_{s+1}, s)$ holds. This guarantees that X_i and Y have no upper bound in the K-degrees, for each $i \le k$. We also require the inductive assumption that $\mu(\{Z \in 2^\omega : Z \succ \sigma_s\} \cap \mathcal{R}) > 0$. This ensures that $Y \in \mathcal{R}$

because $\mathcal{R}$ is closed. Therefore, Y is 1-random. Finally, the construction will be done relative to the oracle $X_0 \oplus \cdots \oplus X_k \oplus \emptyset'$ to guarantee that $Y \leq_T X_0 \oplus \cdots \oplus X_k \oplus \emptyset'$.

Stage $s = 0$: Let $\sigma_0 = \emptyset$. Note that $\mu(\{Z \in 2^\omega : Z \succ \sigma_0\} \cap \mathcal{R}) = \mu\mathcal{R} \geq 1/2 > 0$, so the inductive assumption holds for the base case.

Stage $s + 1$: We have constructed σ_s such that $\mu(\{Z \in 2^\omega : Z \succ \sigma_s\} \cap \mathcal{R}) > 0$. Using the oracle $X_0 \oplus \cdots \oplus X_k \oplus \emptyset'$, search for $\tau \succ \sigma_s$ and $p \in \mathbb{Q}$ such that $B(\tau, s)$, $p < 2^{-|\tau|}$ and $\neg A(\tau, p)$. If these are found, then set $\sigma_{s+1} = \tau$ and note that it satisfies our requirements. In particular, $\mu(\{Z \in 2^\omega : Z \succ \sigma_{s+1}\} \cap \mathcal{R}) \geq 2^{-|\sigma_{s+1}|} - p > 0$. All that remains is to verify that the search succeeds. We know by Corollary 5.3 (ii) that if

$$\mathcal{G} = \{Z \in 2^\omega : (\forall i \leq k)\ X_i \text{ and } Z \text{ have no upper bound in the } K\text{-degrees}\},$$

then $\mu\mathcal{G} = 1$. Therefore, $\mu(\mathcal{G} \cap [\sigma_s] \cap \mathcal{R}) > 0$. There is a $Z \in \mathcal{G} \cap [\sigma_s] \cap \mathcal{R}$ such that $\mu([Z \upharpoonright n] \cap \mathcal{R}) > 0$, for all $n \in \omega$. Otherwise, $\mathcal{G} \cap [\sigma_s] \cap \mathcal{R}$ could be covered with a countable collection of measure zero sets. Because $Z \in \mathcal{G}$, there is an $n > |\sigma_s|$ such that $B(Z \upharpoonright n, s)$. Letting $\tau = Z \upharpoonright n$ ensures that $\tau \succ \sigma_s$, $B(\tau, s)$ and $\mu([\tau] \cap \mathcal{R}) > 0$. The last condition implies that there is a rational $p < 2^{-|\tau|}$ such that $\neg A(\tau, p)$.

This completes the construction. $\square$

7. Plain complexity and randomness

It turns out that much of the information implicit in the prefix-free complexity of the initial segments of a real can also be determined from the plain complexity of those initial segments. There is substance to this claim; it was not even known that 1-randomness can be characterized in terms of initial segment C-complexity. In Theorem 7.1 we give such characterizations. We prove that $X \in 2^\omega$ is 1-random iff $(\forall n)\ C(X \upharpoonright n) \geq n - K(n) - O(1)$. Although this is a natural equivalence, it falls somewhat short of giving a plain Kolmogorov complexity characterization of 1-randomness. A somewhat more satisfying solution to that problem is also provided by Theorem 7.1: X is 1-random iff for every computable $g : \omega \to \omega$ such that $\sum_{n\in\omega} 2^{-g(n)}$ is finite, $(\forall n)\ C(X \upharpoonright n) \geq n - g(n) - O(1)$.

Combining Theorems 5.1 and 7.1, we prove that $\leq_C$ implies $\leq_{vL}$. One consequences is that n-randomness is a C-degree invariant for every $n \in \omega$. As was mentioned in the introduction, 2-randomness is already known to have a C-complexity characterization [20, 23]: X is 2-random iff $(\exists^\infty n)\ C(X \upharpoonright n) \geq n - O(1)$.

Solovay [28, section V] constructed a computable function $h : \omega \to \omega$ such that $\sum_{n\in\omega} 2^{-h(n)} \leq \infty$ but $(\exists^\infty n)\ h(n) \leq K(n) + O(1)$. We use a specific example of such a function in this section. Define a computable function $G : \omega \to \omega$ by

$$G(n) = \begin{cases} K_{s+1}(t), & \text{if } n = 2^{\langle s,t\rangle} \text{ and } K_{s+1}(t) \neq K_s(t) \\ n, & \text{otherwise.} \end{cases}$$

Note that $\sum_{n\in\omega} 2^{-G(n)} \leq \sum_{n\in\omega} 2^{-n} + \sum_{t\in\omega} \sum_{m\geq K(t)} 2^{-m} = 2 + 2\sum_{t\in\omega} 2^{-K(t)} < \infty$. We show that either $n - K(n)$ or $n - G(n)$ can be taken as the cutoff between the initial segment plain complexity of 1-random and non-1-random reals.

Theorem 7.1. *For $X \in 2^\omega$, the following are equivalent:*

(i) *X is 1-random.*
(ii) $(\forall n)\ C(X \upharpoonright n) \geq n - K(n) - O(1)$.

(iii) $(\forall n)\ C(X \upharpoonright n) \geq n - g(n) - O(1)$, *for every computable* $g\colon \omega \to \omega$ *such that* $\sum_{n\in\omega} 2^{-g(n)}$ *is finite.*
(iv) $(\forall n)\ C(X \upharpoonright n) \geq n - G(n) - O(1)$.

Condition (ii) of Theorem 7.1 is similar to a characterization that Gács [9] gave of 1-randomness in terms of *length conditional* Kolmogorov complexity. He proved that $X \in 2^\omega$ is 1-random iff $(\forall n)\ C(X \upharpoonright n \mid n) \geq n - K(n) - O(1)$, where $C(X \upharpoonright n \mid n)$ denotes the *Kolmogorov complexity of* $X \upharpoonright n$ *given* n (see [17] for a definition). Because $C(X \upharpoonright n) \geq C(X \upharpoonright n \mid n) - O(1)$, for all $X \in 2^\omega$ and $n \in \omega$, Gács' condition implies condition (ii) of the theorem. Therefore, the Gács characterization proves that (i) implies (ii). We give another proof of this implication below.

First, we prove the most difficult part of Theorem 7.1.

Lemma 7.2. *If* $(\forall n)\ C(X \upharpoonright n) \geq n - G(n) - O(1)$, *then* $X \in 2^\omega$ *is* 1*-random.*

Proof. By Lemma 2.3 (ii), there a $c \in \omega$ such that

$$|\{\tau \in 2^t \colon K(\tau) \leq t - k\}| \leq 2^{t-K(t)-k+c},$$

for all $t, k \in \omega$. We construct a partial computable (non-prefix-free) function $M\colon 2^{<\omega} \to 2^{<\omega}$. For $s, t \in \omega$, let $n = 2^{\langle s,t\rangle}$. To $\langle s, t\rangle$ we devote the M-programs with lengths from $n/2 + c + 1$ to $n + c$. Note that distinct pairs do not compete for elements in the domain of M. For $k \in \omega$, let $m = n - K_{s+1}(t) - k + c$. Clearly, $m \leq n + c$. If $m \geq n/2 + c + 1$, then for every $\sigma \in 2^n$ such that $K(\sigma \upharpoonright t) \leq t - k$, try to give σ an M-program of length m. Different k do not compete for programs, but it is still possible that there are not enough strings of length m for all such σ. However, this cannot happen if $K_s(t) = K(t)$. This is because the number of $\sigma \in 2^n$ for which $K(\sigma \upharpoonright t) \leq t - k$ is bounded above by $2^{t-K(t)-k+c}2^{n-t} = 2^{n-K(t)-k+c} = 2^m$, so there is enough room in the domain of M to handle every such σ. This completes the construction of M.

Assume that $X \in 2^\omega$ is not 1-random. For each $k \in \omega$, there is an $t \in \omega$ such that $K(X \upharpoonright t) \leq t - k$ and t is large enough that $K(t) \leq 2^{t-1} - k - 1$. Take the least $s \in \omega$ such that $K_{s+1}(t) = K(t)$ and let $n = 2^{\langle s,t\rangle}$. Then

$$m = n - K(t) - k + c \geq n - 2^{t-1} + k + 1 - k - c \geq n/2 + c + 1,$$

because $n = 2^{\langle s,t\rangle} \geq 2^t$. This implies that there is an M-program for $X \upharpoonright n$ of length $m = n - K(t) - k + c$. Also note that $G(n) = K_{s+1}(t) = K(t)$. So,

$$C(X \upharpoonright n) \leq C_M(X \upharpoonright n) + O(1) \leq n - K(t) - k + c + O(1) \leq n - G(n) - k + O(1),$$

where the constant is independent of X, n and k. Because k is arbitrary,

$$\liminf_{n\to\infty} C(X \upharpoonright n) - n + G(n) = -\infty.$$

Therefore, if $(\forall n)\ C(X \upharpoonright n) \geq n - G(n) - O(1)$, then X is 1-random. This completes the proof. $\square$

Proof of Theorem 7.1. (i) $\Longrightarrow$ (ii): Define

$$\mathcal{I}_k = \{X \in 2^\omega \colon (\exists n)\ C(X \upharpoonright n) < n - K(n) - k\}.$$

As usual, let K_s and C_s denote the approximations to K and C at stage s. Then $(\exists n)(\exists s)\ C_s(X \upharpoonright n) + K_s(n) < n - k$ iff $X \in \mathcal{I}_k$. Therefore, $\mathcal{I}_k$ is a Σ^0_1 class.

Fewer than $2^{n-K(n)-k}$ V-programs have length less than $n-K(n)-k$, so $|\{\sigma \in 2^n : C(\sigma) < n-K(n)-k\}| \le 2^{n-K(n)-k}$. Therefore,

$$\mu\mathcal{I}_k \le \sum_{n\in\omega} \mu\{X \in 2^\omega : C(X\restriction n) < n-K(n)-k\}$$
$$\le \sum_{n\in\omega} 2^{-n}2^{n-K(n)-k} = 2^{-k}\sum_{n\in\omega} 2^{-K(n)} \le 2^{-k}.$$

So, $\{\mathcal{I}_k\}_{k\in\omega}$ is a Martin-Löf test. If X is 1-random, then $X \notin \mathcal{I}_k$ for large enough k. In other words, $(\forall n)\ C(X\restriction n) \ge n-K(n)-k$.

(ii) $\implies$ (iii): Let $g\colon \omega \to \omega$ be a computable function such that $\sum_{n\in\omega} 2^{-g(n)} < \infty$. By the minimality of K as an information content measure, $(\forall n)\ K(n) \le g(n)+O(1)$. Therefore, if $(\forall n)\ C(X\restriction n) \ge n-K(n)-O(1)$, then $(\forall n)\ C(X\restriction n) \ge n-g(n)-O(1)$.

(iii) $\implies$ (iv) is immediate because G is computable and $\sum_{n\in\omega} 2^{-G(n)}$ is finite.

Finally, (iv) $\implies$ (i) was proved in Lemma 7.2. □

As with K-complexity, the C-complexity of the initial segments of a real determines its vL-degree. This is a consequence of the following result.

Theorem 7.3. *Assume that $Z \in 2^\omega$ is 1-random. The following are equivalent:*
 (i) *X is 1-Z-random.*
 (ii) $(\forall n)\ C(X\restriction n) \ge n-K^Z(n)-O(1)$.
 (iii) $(\forall n)\ C(X\restriction n)+K(Z\restriction n) \ge 2n-O(1)$.

The following lemma is folklore (see [17, page 138]).

Lemma 7.4. *For any real $X \in 2^\omega$, $(\forall n)\ C(X\restriction(X\restriction n)) \le X\restriction n - n + O(1)$.*

Proof of Theorem 7.3. (i) $\implies$ (ii): Suppose that Z is 1-random. If X is 1-Z-random, then by relativizing Theorem 7.1,

$$(\forall n)\ C(X\restriction n) \ge C^Z(X\restriction n)-O(1) \ge n-K^Z(n)-O(1).$$

(ii) $\implies$ (iii): Since Z is 1-random, the ample excess lemma gives $K^Z(n) \le K(Z\restriction n)-n+O(1)$, for all $n\in\omega$. So $(\forall n)\ C(X\restriction n) \ge n-K^Z(n)-O(1) \ge 2n-K(Z\restriction n)-O(1)$.

(iii) $\implies$ (i): By Lemma 7.4, $(\forall n)\ K(Z\restriction(X\restriction n)) \ge 2\cdot X\restriction n - C(X\restriction(X\restriction n)) - O(1) \ge 2\cdot X\restriction n - X\restriction n + n - O(1) \ge X\restriction n + n - O(1)$. By Theorem 5.1, $Z\oplus X$ is 1-random. So, by Theorem 3.1, X is 1-Z-random. □

Note that assuming (iii), we have $K(Z\restriction n) \ge 2n - C(X\restriction n) - O(1) \ge n - O(1)$ for all $n\in\omega$, so Z is 1-random. This gives us a cleaner way of expressing the equivalence of (i) and (iii).

Corollary 7.5. *$X\oplus Z$ is 1-random iff $(\forall n)\ C(X\restriction n)+K(Z\restriction n) \ge 2n-O(1)$.*

An immediate consequence that the C-degrees refine the vL-degrees.

Corollary 7.6. $X \le_C Y \implies X \le_{vL} Y$.

Therefore, the conclusions of Corollary 5.3 hold for the C-degrees as well.

Corollary 7.7.
 (i) *If $X \le_C Y$ and X is n-random, then Y is n-random.*

(ii) *If $X \oplus Y$ is 1-random, then $X \mid_C Y$ and X and Y have no upper bound in the C-degrees. Therefore, there is no join in the C-degrees.*
(iii) *If $m \neq n$, then $\Omega^{\emptyset^{(n)}}$ and $\Omega^{\emptyset^{(m)}}$ have no upper bound in the C-degrees.*
(iv) *If $\Omega = \bigoplus_{n\in\omega} \Omega_n$, then $\{\Omega_n\}_{n\in\omega}$ is a C-antichain of Δ^0_2 1-random reals (and again, no two have an upper bound in the C-degrees).*

Finally, we remark that it requires only superficial modification to the proof of Proposition 6.3 to prove the corresponding result for the C-degrees: if $X_0, \dots, X_k$ are 1-random reals, then there is a 1-random real $Y \leq_T X_0 \oplus \cdots \oplus X_k \oplus \emptyset'$ such that, for every $i \leq k$, Y and X_i have no upper bound in the C-degrees. This implies that there is a Δ^0_2 1-random real that is not C-above Ω, hence $\leq_{vL}$ and $\leq_C$ differ on the 1-random reals.

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