Computing descending sequences in linear orderings

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Online Seminar on Computability Theory and Applications
August 2020
How hard is it to compute an infinite descending sequence in an ill-founded linear ordering $L$?

This problem reduces to the problem of computing a path on an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$:

- Given $L$, compute the tree $T(L)$ of all finite descending sequences in $L$;
- Every path $P$ on $T(L)$ computes an infinite descending sequence $S$ in $L$. 

\[ L \xrightarrow{\text{computable}} T(L) \]

\[ S \xleftarrow{\text{computable}} P \]
Going in the opposite direction

- Given a tree $T$, we can compute its **Kleene-Brouwer ordering** $\text{KB}(T)$, defined by $\sigma \leq_{\text{KB}} \tau$ iff $\sigma$ extends $\tau$ or is lexicographically below $\tau$.

- $\text{KB}(T)$ is ill-founded if and only if $T$ is ill-founded.

- Given a descending sequence $(\sigma_i)_{i \in \mathbb{N}}$ in $\text{KB}(T)$,

  $$P(n) = \lim_{i \to \infty} \sigma_i(n)$$

  is a path on $T$.

\[ T \xrightarrow{\text{computable}} \text{KB}(T) \]

\[ \downarrow \text{limit-computable} \]

\[ P \xleftarrow{\text{computable}} S \]

Can we do better?
Weihrauch reducibility: represented spaces

If each object in a space $X$ can be “encoded” as a real, then we can make it into a represented space, thereby transferring notions of computability from $\mathbb{N}^\mathbb{N}$ to $X$.

Formally, a represented space is a pair $(X, \delta)$ where $\delta : \subseteq \mathbb{N}^\mathbb{N} \to X$ is a (possibly partial) surjection.

Each element of $X$ is named by some (possibly multiple) $p \in \mathbb{N}^\mathbb{N}$ via $\delta$.

Examples:

$$\mathbb{N}^\mathbb{N}, \mathbb{N}, \text{LO}, \text{Tr}, \Pi^1_1(\mathbb{N}), \Sigma^1_1(\text{LO})$$

All of the above spaces can be represented in standard ways.
Weihrauch reducibility: problems

Examples of problems

**DS:** given an ill-founded linear ordering, produce any infinite descending sequence

**\(C_{\mathbb{N}^\mathbb{N}}:**** given an ill-founded subtree of \(\mathbb{N}^{<\mathbb{N}}\), produce any path

**\(\text{lim}:**** given a convergent sequence of reals, produce its limit

Formally, a problem \(f : \subseteq X \Rightarrow Y\) is a (possibly partial) multivalued function between represented spaces.

We also think of a problem as a set of instance-solution pairs:

- If \(x \in \text{dom}(f)\) then we say that \(x\) is an \(f\)-instance.
- For each \(f\)-instance \(x\), the set of \(f\)-solutions to \(x\) is \(f(x) \subseteq Y\).
Weihrauch reducibility $\leq_W$

**Definition**

A problem $f$ is **Weihrauch reducible** to a problem $g$ ($f \leq_W g$) if there are computable functions $\Phi, \Psi : \subseteq \mathbb{N}^\mathbb{N} \rightarrow \mathbb{N}^\mathbb{N}$ such that:

- if $p$ is a name for an $f$-instance, then $\Phi(p)$ is a name for a $g$-instance;
- if $p$ is a name for an $f$-instance and $q$ is a name for a $g$-solution to $\Phi(p)$, then $\Psi(p, q)$ is a name for an $f$-solution to $p$.

We call $\Phi$ and $\Psi$ **forward** and **backward** functionals respectively.

\[
\begin{array}{c}
\text{$f$-instance $p$} \xrightarrow{\Phi(\cdot)} \text{$g$-instance $\Phi(p)$} \\
\downarrow \quad \downarrow \\
\text{$f$-solution $\Psi(p, q)$} \xleftarrow{\Psi(p, \cdot)} \text{$g$-solution $q$}
\end{array}
\]
Our first reduction shows that $\text{DS} \leq_{\text{W}} \text{C}_N^N$.

Our second reduction is not a Weihrauch reduction from $\text{C}_N^N$ to $\text{DS}$ because our paths are obtained by applying the limit to descending sequences in $\text{KB}(T)$. Nonetheless:

**Proposition**

$\text{C}_N^N \equiv_{\text{W}} \text{lim} \ast \text{DS}$.

$f \ast g$ is the compositional product (Brattka, Gherardi, Marcone), which captures what can be achieved by first applying $g$, followed by some computation, and then applying $f$.

**Question (which we will answer)**

Do we have $\text{C}_N^N \equiv_{\text{W}} \text{DS}$? Equivalently, does $\text{C}_N^N \leq_{\text{W}} \text{DS}$?
Our results: $\mathbb{C}_{\mathbb{N}\mathbb{N}} \preceq_W \text{DS (and more)}$

We show that DS is quite weak in terms of uniform computational strength.

**Theorem (G., Pauly, Valenti)**

A single-valued problem is Weihrauch reducible to DS if and only if it is Weihrauch reducible to $\text{lim}$, i.e.,

$$\max_{\leq W} \{ f_0 : \subseteq \mathbb{Z} \to \mathbb{N}^\mathbb{N} \mid f_0 \leq_W \text{DS} \} \equiv_W \text{lim}.$$

**Corollary**

The following problems are not Weihrauch reducible to DS: LPO$'$, ADS, lim$'$, UC$_{\mathbb{N}\mathbb{N}}$, C$_{\mathbb{N}\mathbb{N}}$.

**Open question**

Is KL (König’s lemma) Weihrauch reducible to DS?
Our results: first-order part of DS

Our techniques characterize the problems which have codomain $\mathbb{N}$ and are reducible to DS:

**Definition**

Let $\Pi^1_1$-Bound : $\subseteq \Pi^1_1(\mathbb{N}) \Rightarrow \mathbb{N}$ be the following problem: given a $\Pi^1_1$-code for a finite subset of $\mathbb{N}$, produce a bound.

**Theorem (G., Pauly, Valenti)**

$max_{\leq W} \{ f_0 : \subseteq \mathbb{Z} \Rightarrow \mathbb{N} \mid f_0 \leq_W DS \} \equiv_W \Pi^1_1$-Bound.

Dzhafarov, Solomon, Yokoyama (ta) were the first to define and study the first-order part $^1f$ of an arbitrary problem $f$:

$^1f \equiv_W max_{\leq W} \{ f_0 : \subseteq \mathbb{Z} \Rightarrow \mathbb{N} \mid f_0 \leq_W f \}$. 
Proof that $^1\text{DS} \leq_W \Pi^1_1\text{-Bound}$

Suppose $f : \subseteq \mathbb{Z} \Rightarrow \mathbb{N}$ reduces to DS. Given an $f$-instance $p$, we can find an $f$-solution to $p$ as follows.

1. At stage $s$, we can compute a finite piece $L_s$ of the DS-instance defined by the forward functional.
2. List all descending sequences in $L_s$ on which the backward functional converges (and hence gives a potential $f$-solution).
3. If such descending sequences exist, we can guess an $f$-solution by picking the $L_s$-rightmost descending sequence $F_s$.
4. The set of $s$ such that $F_s$ is undefined or not extendible is $\Pi^1_1,p$.
5. Apply $\Pi^1_1\text{-Bound}$ to obtain an $s$ such that $F_s$ is extendible. This yields an $f$-solution to $p$.

Hence $f \leq_W \Pi^1_1\text{-Bound}$. 
\( \Pi^1_1 \)-Bound and \( \Sigma^1_1 \) choice principles

\( \Sigma^1_1 \)-C\(_N\): given a \( \Sigma^1_1 \)-code for a nonempty subset of \( \mathbb{N} \), produce an element of the set.

\( \Sigma^1_1 \)-C\(_{\text{cof}}\): given a \( \Sigma^1_1 \)-code for a cofinite subset of \( \mathbb{N} \), produce an element of the set.

It is easy to see that \( \Sigma^1_1 \)-C\(_{\text{cof}}\) is Weihrauch equivalent to \( \Pi^1_1 \)-Bound.

**Theorem (Angles d’Auriac, Kihara ta)**

\[ \overline{\Sigma^1_1 \text{-C}_{\text{cof}}} <_W \overline{\Sigma^1_1 \text{-C}}_{\text{N}}, \text{ hence } \overline{\Sigma^1_1 \text{-C}_{\text{cof}}} <_W \overline{\Sigma^1_1 \text{-C}}_{\text{N}}. \]

It is easy to see that the first-order part of \( C_{\mathbb{N}^\mathbb{N}} \) is \( \Sigma^1_1 \)-C\(_N\), so this theorem and our results imply that DS and \( C_{\mathbb{N}^\mathbb{N}} \) can be separated by considering their first-order parts.

In fact our results imply that there is a single-valued problem with codomain 2 which separates DS and \( C_{\mathbb{N}^\mathbb{N}} \), namely LPO′.
Kihara, Marcone, Pauly asked if $\Sigma^1_1$-$C_N <_W C_{NN}$.

**Theorem (Angles d’Auriac, Kihara ta)**

$\Sigma^1_1$-$C_N <_W C_{NN}$. In fact $\text{ATR}_2 \not<_W \Sigma^1_1$-$C_N$.

AK proved the above separation using a pair of inseparable $\Pi^1_1$ sets. We will extend their techniques to prove a stronger result about a strengthening of DS.

**Definition (G.)**

Let $\text{ATR}_2 : \text{LO} \Rightarrow \mathbb{N}^\mathbb{N}$ be the following problem: given a linear ordering $L$, produce either an infinite descending sequence in $L$ or a jump hierarchy on $L$ (with a bit indicating which type of solution we produce).

In fact it suffices to consider $\text{ATR}_2$ restricted to computable linear orderings.
Our earlier results imply that $\text{ATR}_2 \not\leq_W \text{DS}$, but much more is true:

**Definition**

Let $\Sigma^1_1$-DS : $\subseteq \Sigma^1_1(\text{LO}) \Rightarrow \mathbb{N}^\mathbb{N}$ be the following problem: given a $\Sigma^1_1$-code for an ill-founded linear ordering, produce an infinite descending sequence.

**Theorem (G., Pauly, Valenti)**

$\text{ATR}_2 \not\leq_W \Sigma^1_1$-DS, hence $\Sigma^1_1$-DS $\subsetneq_W C_{\mathbb{N}^\mathbb{N}}$.

This means that $C_{\mathbb{N}^\mathbb{N}}$ does not reduce to DS even if we allow the forward functional to be $\Sigma^1_1$ rather than computable.

On the other hand, we saw earlier that if we allow the backward functional to be limit-computable, then $C_{\mathbb{N}^\mathbb{N}}$ reduces to DS.
We can think of ATR$_2$ as a “union” of DS and JH: given a linear ordering which supports a jump hierarchy, produce a jump hierarchy.

It is well known that the set of indices of ill-founded linear orderings is $\Sigma^1_1$-complete.

Harrington (unpublished) showed that the set of indices of linear orderings which support a jump hierarchy is also $\Sigma^1_1$-complete.

**Theorem (G., generalizing Harrington’s proof)**

Any $\Sigma^1_1$ set which separates $\text{wf}$ and $\text{hds}$ is $\Sigma^1_1$-complete.

($\text{wf}$ and $\text{hds}$ are the set of indices for well-founded linear orderings and linear orderings with hyp descending sequences respectively.)

This generalizes Harrington’s result because if a computable linear ordering has a hyp descending sequence, then it does not support a jump hierarchy (Friedman).
ATR\textsubscript{2} and inseparable $\Pi_1^1$ sets

**Theorem (G., generalizing Harrington’s proof)**

Any $\Sigma_1^1$ set which separates $\text{wf}$ and $\text{hds}$ is $\Sigma_1^1$-complete.

($\text{wf}$ and $\text{hds}$ are the set of indices for well-founded linear orderings and linear orderings with hyp descending sequences respectively.)

By $\Sigma_1^1$-separation,

**Corollary**

$\text{wf}$ and $\text{hds}$ cannot be separated by disjoint $\Sigma_1^1$ sets.

Angles d’Auriac, Kihara used the corollary to prove that

$\text{ATR}_2 \not\leq_W \Sigma_1^1$-$\text{C}_N$.

We will use the corollary to prove that $\text{ATR}_2 \not\leq_W \Sigma_1^1$-$\text{DS}$.  

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Proof that $\text{ATR}_2 \not\leq_W \Sigma^1_1\text{-DS}$ (G., Pauly, Valenti)

Suppose that $\text{ATR}_2 \leq_W \Sigma^1_1\text{-DS}$. For each computable linear ordering $L_e$, the forward functional produces a $\Sigma^1_1$-code $\Phi(L_e)$ for an ill-founded linear ordering.

For the same $e$, the backward functional may produce either descending sequences in $L_e$ or jump hierarchies on $L_e$, depending on which descending sequence in $\Sigma^1_1\text{-DS}(\Phi(L_e))$ is given. However, descending sequences are sufficiently homogeneous so

**Lemma**

*For each $e$, either $\text{DS}(L_e)$ or $\text{JH}(L_e)$ is Muchnik reducible to $\Sigma^1_1\text{-DS}(\Phi(L_e))$.***

Then we have disjoint $\Sigma^1_1$ sets which separate $\text{wf}$ and $\text{hds}$:

$\text{wf} \subseteq \{ e \in \mathbb{N} : \text{DS}(L_e) \text{ is not Muchnik reducible to } \Sigma^1_1\text{-DS}(\Phi(L_e)) \}$

$\text{hds} \subseteq \{ e \in \mathbb{N} : \text{JH}(L_e) \text{ is not Muchnik reducible to } \Sigma^1_1\text{-DS}(\Phi(L_e)) \}$.

This contradicts the corollary on the previous slide. Thank you!