Redundancy of information: Lowering effective dimension

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Summary

We study the interaction between effective Hausdorff dimension

\[
\dim(X) = \lim_{n \to \infty} \inf \frac{K(X | n)}{n} \in [0, 1]
\]

and Besicovitch pseudo-distance

\[
d(X, Y) = \lim_{n \to \infty} \sup \frac{|(X | n) \Delta (Y | n)|}{n} \in [0, 1]
\]

of binary sequences. Specifically, fix \( t < s \) in \([0, 1]\).

- Given \( X \) with \( \dim(X) = t \), how close to \( X \) can we find \( Y \) with \( \dim(Y) = s \)?

- Given \( Y \) with \( \dim(Y) = s \), how close to \( Y \) can we find \( X \) with \( \dim(X) = t \)?

This line of inquiry was initiated by Greenberg, Miller, Shen, Westrick (henceforth GrMShW). We continue their work.
The Kolmogorov complexity $K(\sigma)$ of a finite binary string $\sigma$ is the length of the shortest description of $\sigma$, where descriptions are given by a fixed universal Turing machine.

We are concerned with the asymptotics of $\frac{K(\sigma)}{|\sigma|}$ (where $\sigma$ is an initial segment of some $X \in 2^\omega$), so it does not matter which universal Turing machine we fix.

Nor does it matter whether we use plain Kolmogorov complexity or prefix-free Kolmogorov complexity.
The entropy function $H : [0, 1] \rightarrow [0, 1]$

Given a string $\sigma$ of length $n$, here is a way to describe it:

1. specify the number of 1s and 0s in $\sigma$ (say $pn$ and $(1 - p)n$ respectively); and

2. specify $\sigma$ among the strings of length $n$ with $pn$ many 1s.

(1) can be done with $O(\log n)$ bits.

(2) can be done with $H(p)n$ bits, where

$$H(p) = -p \log(p) - (1 - p) \log(1 - p)$$

is the entropy function.
Effective Hausdorff dimension of sequences

Definition (Lutz; Mayordomo)
The (effective Hausdorff) dimension of a sequence $X \in 2^\omega$ is

$$\dim(X) = \lim_{n \to \infty} \inf \frac{K(X \upharpoonright n)}{n} \in [0, 1].$$

Observations:

▶ Computable sequences have dimension 0.
▶ Martin-Löf random sequences have dimension 1.
▶ Flipping every bit in a sequence does not change its dimension.
Upper density and dimension

If a sequence $X$ has upper density $p$, i.e.,

$$\limsup_{n \to \infty} \frac{|\{i < n : X(i) = 1\}|}{n} = p,$$

then we can bound the dimension of $X$ in terms of $p$:

**Proposition**

A sequence with upper density $p$ has dimension $\leq H(p)$.

**Corollary**

If a sequence has dimension $s$, then its upper density is at least $H^{-1}(s)$. (We use the branch $H^{-1} : [0, 1] \to [0, 1/2]$.)
Hamming distance and Besicovitch pseudo-distance

The Hamming distance $\Delta(\sigma, \tau)$ between strings $\sigma, \tau \in 2^n$ is the number of bits where they differ.

Definition
The (Besicovitch pseudo-)distance between sequences $X, Y \in 2^\omega$ is

$$d(X, Y) = \limsup_{n \to \infty} \frac{\Delta(X \upharpoonright n, Y \upharpoonright n)}{n} \in [0, 1].$$

Observations:
- The distance between $X$ and $00 \cdots$ is the upper density of $X$.
- If we modify $X$ on a set of positions of upper density 0, then the result $Y$ satisfies $d(X, Y) = 0$. 
Distance versus dimension

Proposition (GrMShW)
If \( \dim(X) = t \) and \( \dim(Y) = s \), then \( |s - t| \leq H(d(X, Y)) \).

In particular:
1. The previous proposition is the special case where \( Y \) is 00···.
2. If \( d(X, Y) = 0 \), then \( X \) and \( Y \) have the same dimension.

Proof idea: We can describe an initial segment of \( X \) by describing
the corresponding initial segment of \( Y \), as well as their differences.
This shows that
\[
t \leq s + H(d(X, Y)).
\]
Distance versus dimension

Proposition (GrMShW)
If \( \dim(X) = t \) and \( \dim(Y) = s \), then \( |s - t| \leq H(d(X, Y)) \), i.e.,

\[
d(X, Y) \geq H^{-1}(|s - t|).
\]

Motivating Question
Is this the best possible bound?

In a weak sense, yes:

Proposition (GrMShW)
For every \( t < s \), there are \( X \) and \( Y \) with \( \dim(X) = t \), \( \dim(Y) = s \), and \( d(X, Y) \leq H^{-1}(s - t) \) (hence \( d(X, Y) = H^{-1}(s - t) \)).

However, it is not the case that for every \( X \) of dimension \( t \), there is some \( Y \) of dimension \( s \) such that \( d(X, Y) \leq H^{-1}(s - t) \).
Increasing dimension (from $t$ to $s$)

Observation (GrMShW)

Suppose $0 < t < s$. There is some $X$ of dimension $t$ such that for every $Y$ of dimension $s$, $d(X, Y) > H^{-1}(s - t)$.

To see this, fix $X$ with dimension $t$ and density $H^{-1}(t)$. For every $Y$ with dimension $s$, the density of $Y$ is at least $H^{-1}(s)$, so

$$d(X, Y) \geq H^{-1}(s) - H^{-1}(t) > H^{-1}(s - t).$$
Increasing dimension (from $t$ to $s$)

**Observation (GrMShW)**

Fix $X$ with dimension $t$ and density $H^{-1}(t)$. For every $Y$ with dimension $s$, the density of $Y$ is at least $H^{-1}(s)$, so

$$d(X, Y) \geq H^{-1}(s) - H^{-1}(t).$$

The above is the worst that could happen when trying to increase the dimension of a given sequence $X$:

**Theorem (GrMShW)**

Suppose $t < s$. For every $X$ of dimension $t$, there is some $Y$ of dimension $s$ such that $d(X, Y) \leq H^{-1}(s) - H^{-1}(t)$. 
Lowering dimension (from $s$ to $t$)

Given $Y$ of dimension $s$, how close to $Y$ can we find some $X$ of dimension $t$?

$H^{-1}(s - t)$ is the closest that we can hope for, but this is not always attainable.

An issue arises if the information in $Y$ is stored redundantly (so it is harder to erase).
Lowering dimension (from $s$ to $t$): Redundancy in $Y$

(GrMShW) Take $Y$ to be $Z \oplus Z$, where $Z$ is a random.

Imagine you’re trying to flip bits of $Y$ in order to obtain an $X$ of lower dimension.

In order for you to succeed, it must be hard to recover $Y$ from $X$.

$X$ can detect (for free) its inconsistencies, i.e., the $i$ such that $X(2i) \neq X(2i + 1)$. It is relatively cheap to fix all inconsistencies. Example:

\[
\begin{align*}
X & \quad 0000110100101101 \cdots \\
\text{Extra info} & \quad 001 \cdots \\
\tilde{X} & \quad 0000110000001111 \cdots
\end{align*}
\]

If, in addition to the above, we specify the set of $i$ such that $X(2i) = X(2i + 1) \neq Z(i)$, then we can recover all of $Y$. 
Lowering dimension (from $s$ to $t$)

**Theorem (GrMShW)**
For each $Y$ of dimension $s$ and each $t < s$, there is some $X$ of dimension $t$ with $d(X, Y) \leq H^{-1}(1 - t)$.

This was proved using the corresponding result for strings:

**Proposition (GrMShW)**
For each $\sigma \in 2^n$ and $t \in [0, 1]$, there is some $\tau \in 2^n$ such that

\[
\frac{K(\tau)}{n} \leq t + O(\log n/n)
\]

\[
\frac{\Delta(\sigma, \tau)}{n} \leq H^{-1}(1 - t).
\]

If $s = 1$, the above theorem yields an optimal result.
Lowering dimension (from $s < 1$ to $t$): Another strategy

If $s < 1$, there is another strategy for finding a nearby $X$ of dimension $t$.

The previous theorem was proved by applying the previous proposition to each interval in $Y$ to obtain $X$. Instead:

- We leave some intervals in $Y$ unchanged, and
- apply the previous proposition to the other intervals to obtain strings of dimension $< t$.

If $t$ is sufficiently close to $s$, then this strategy is better.
Lowering dimension (from $s$ to $t$)

**Theorem (GoMSoW)**

For each $Y$ of dimension $s$ and each $t < s$, there is some $X$ of dimension $t$ such that

$$d(X, Y) \leq \begin{cases} H^{-1}(1 - t) & \text{if } t \leq 1 - H(2^{s-1}) \\ \frac{s-t}{-\log(2^{1-s} - 1)} & \text{otherwise} \end{cases}.$$

**Observations:**

1. For $s = 1$, this specializes to the previous theorem of GrMShW.

2. The above piecewise function is continuous, and even differentiable.

**Corollary (GoMSoW)**

For each $Y$ of dimension $s$ and every $\epsilon > 0$, there is some $t < s$ and some $X$ of dimension $t$ such that $d(X, Y) \leq \epsilon$. 