Topological optimization of rod-stirring devices

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The taffy puller

[Photo and movie by M. D. Finn.]

[Movie 1]
The mixograph

Model experiment for kneading bread dough:

[Department of Food Science, University of Wisconsin. Photos by J-LT.]
Planetary mixers

In food processing, rods are often used for stirring.
Experiment of Boyland, Aref & Stremler

[movie 3]  [movie 4]

The three rods of the taffy puller in a space-time diagram. Defines a braid on \( n = 3 \) strands, \( \sigma_1^2 \sigma_2^{-2} \) (three periods shown).
Braid description of mixograph

\[ \sigma_3 \sigma_2 \sigma_3 \sigma_5 \sigma_6^{-1} \sigma_2 \sigma_3 \sigma_4 \sigma_3 \sigma_1^{-1} \sigma_2^{-1} \sigma_5 \]

braid on \( B_7 \), the braid group on 7 strands.
Topological entropy of a braid

**Burau representation** for 3-braids:

\[
[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

\[
[\sigma_1^{-1} \sigma_2] = [\sigma_1^{-1}] \cdot [\sigma_2] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.
\]

This matrix has **spectral radius** \((3 + \sqrt{5})/2\) (Golden Ratio\(^2\)), and hence the topological entropy is \(\log[(3 + \sqrt{5})/2]\).

This is the growth rate of a ‘rubber band’ caught on the rods.

This matrix trick only works for 3-braids, unfortunately.
Optimizing over generators

- Entropy can grow without bound as the length of a braid increases;
- A proper definition of optimal entropy requires a cost associated with the braid.
- Divide the entropy by the smallest number of generators required to write the braid word.
- For example, the braid $\sigma_1^{-1} \sigma_2$ has entropy $\log[(3 + \sqrt{5})/2]$ and consists of two generators.
- Its Topological Entropy Per Generator (TEPG) is thus $\frac{1}{2} \log[(3 + \sqrt{5})/2] = \log[\text{Golden Ratio}]$.
- Assume all the generators are used (stronger: irreducible).
Optimal braid

• In $B_3$ and $B_4$, the optimal TEPG is $\log[\text{Golden Ratio}]$. 
• Realized by $\sigma_1^{-1}\sigma_2$ and $\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2$, respectively. 
• In $B_n$, $n > 4$, the optimal TEPG is $< \log[\text{Golden Ratio}]$. 

Why? Recall Burau representation:

$$
[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
$$

Its spectral radius provides a lower bound on entropy. However,

$$
|[\sigma_1]| = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad |[\sigma_2]| = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
$$

provides an upper bound! Need to find Joint Spectral Radius.
Periodic array of rods

- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1 \sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).

![Diagram of periodic lattice of rods](image)

- The entropy per ‘switch’ is $\log(1 + \sqrt{2})$, the Silver Ratio!
- This is optimal for a periodic lattice of two rods (follows from D’Alessandro et al. (1999)).
- Also optimal if we assign cost by simultaneous operation.
Silver mixers

- The designs with entropy given by the Silver Ratio can be realized with simple gears.
- All the rods move at once: very efficient.
Build it!

[movie 6] [movie 7]
Experiment: Silver mixer with four rods
Silver mixer with six rods
Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amount of entropy (stretching of material lines).
- Can optimize to find the best rod motions, but depends on choice of ‘cost function.’
- For two natural cost functions, the Golden Ratio and Silver Ratio pop up!
References


