Simple models of stirring by swimming organisms

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Munk’s Idea

Though it had been mentioned earlier, the first to seriously consider the role of ocean biomixing was Walter Munk (1966):

“...I have attempted, without much success, to interpret [the eddy diffusivity] from a variety of viewpoints: from mixing along the ocean boundaries, from thermodynamic and biological processes, and from internal tides.”
In situ experiments

Katija & Dabiri (2009) looked at jellyfish:

[movie 1] (Palau’s Jellyfish Lake.)
Displacement by a moving body

Maxwell (1869); Darwin (1953); Eames et al. (1994)
A sequence of kicks

Inspired by Einstein’s theory of diffusion (Einstein, 1956): a test particle initially at \( x(0) = 0 \) undergoes \( N \) encounters with an axially-symmetric swimming body:

\[
x(t) = \sum_{k=1}^{N} \Delta L(a_k, b_k) \hat{r}_k
\]

\( \Delta L(a, b) \) is the displacement, \( a_k, b_k \) are impact parameters, and \( \hat{r}_k \) is a direction vector.

\( a > 0, \) but \( b \) can have either sign.
Effective diffusivity

Putting this together,

\[
\langle |x|^2 \rangle = \frac{2Un t}{L} \int \Delta^2_L(a, b) \, da \, db = 4\kappa t, \quad \text{2D}
\]

\[
\langle |x|^2 \rangle = \frac{2\pi Un t}{L} \int \Delta^2_L(a, b) a \, da \, db = 6\kappa t, \quad \text{3D}
\]

which defines the effective diffusivity $\kappa$.

If the number density is low ($nL^d \ll 1$), then encounters are rare and we can use this formula for a collection of particles.
Inviscid cylinders and spheres (treadmill swimmer)

\[ \kappa = \frac{\pi}{3} Un \int a^2 \Delta^2_L(a, b) \, d(\log a) \, d(b/L) \quad \text{3D} \]

Notice \( \Delta_L(a, b) \) is nonzero for \( 0 < b < L \); otherwise independent of \( b \) and \( L \) \( \implies \) have to cross point of closest approach.

\[ a \Delta^2_L(a, b) \text{ (cylinder)} \quad \quad a^2 \Delta^2_L(a, b) \text{ (sphere)} \]
Numerical simulation

- Validate theory using simple simple simulations;
- Large periodic box;
- $N_{\text{swim}}$ swimmers (cylinders of radius 1), initially at random positions, swimming in random direction with constant speed $U = 1$;
- Target particle initially at origin advected by the swimmers;
- Since dilute, superimpose velocities;
- Integrate for some time, compute $|\mathbf{x}(t)|^2$, repeat for a large number $N_{\text{real}}$ of realizations, and average.
A ‘gas’ of swimmers

[movie 2] 100 cylinders, box size = 1000
How well does the dilute theory work?

\[ \frac{\langle |x|^2 \rangle}{2nU^3} \]

- \( n = 10^{-3} \)
- \( n = 5 \times 10^{-4} \)
- \( n = 10^{-4} \)
- Theory
Cloud of particles

[movie 3] (30 cylinders)
Cloud dispersion proceeds by steps

\[ \langle |x|^2 \rangle \]

N = 30

n = 7.5 \times 10^{-4}
Squirmers

Considerable literature on transport due to microorganisms: Wu & Libchaber (2000); Hernandez-Ortiz et al. (2006); Saintillian & Shelley (2007); Ishikawa & Pedley (2007); Underhill et al. (2008); Ishikawa (2009); Leptos et al. (2009)

Lighthill (1952), Blake (1971), and more recently Ishikawa et al. (2006) have considered squirmers:

- Sphere in Stokes flow;
- Steady velocity specified at surface, to mimic cilia;
- Steady swimming condition imposed (no net force on fluid).

(Drescher et al., 2009) (Ishikawa et al., 2006)
Typical squirmer

3D axisymmetric streamfunction for a typical squirmer, in cylindrical coordinates ($\rho, z$):

$$\psi = -\frac{1}{2} \rho^2 + \frac{1}{2r^3} \rho^2 + \frac{3 \beta}{4r^3} \rho^2 z \left( \frac{1}{r^2} - 1 \right)$$

where $r = \sqrt{\rho^2 + z^2}$, $U = 1$, radius of squirmer = 1.

$\beta$ is the amplitude of the stresslet (distinguishes pushers/pullers).

We will use $\beta = 5$ for most of the remainder.
Particle motion for squirmer

A particle near the squirmer’s swimming axis initially (blue) moves towards the squirmer.

After the squirmer has passed the particle follows in the squirmer’s wake.

(The squirmer moves from bottom to top.)

[movie 4]
Squirmer displacements $a^2 \Delta_L^2(a, b)$
Squirmers: Transport
Squirmers: Trajectories

The two peaks in the displacement plot come from ‘incomplete’ trajectories:

For long path length, the effective diffusivity is independent of the swimming path length, and yet the dominant contribution arises from the finiteness of the path (uncorrelated turning directions).
Non-Gaussian PDFs of displacement

- Variance exhibits similar short-time anomalous scaling as in Wu & Libchaber (2000);
- PDF qualitatively matches experiments of Leptos et al. (2009). In our case, exponential tails are due to sticking at the stagnation points on the squirmer’s body.
Conclusions

- Simple **dilute model** works well for a range of swimmers;
- Slip surfaces have an effective diffusivity that is **independent of path length**, for long path length;
- No-slip flows dominated by **sticking** and have a log dependence on path length;

Future work:
- Wake models and turbulence;
- PDF of scalar concentration;
- **Buoyancy effects** for the ocean case;
- Higher densities;
- Schooling: longer length scale?
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