Local and Global Aspects of Mixing

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Experiment of Rothstein et al.: Persistent Pattern

Disordered array of magnets with oscillatory current drive a thin layer of electrolytic solution.

periods 2, 20, 50, 50.5

[Rothstein, Henry, and Gollub, Nature 401, 770 (1999)]
Evolution of Pattern

- “Striations”
- Smoothed by diffusion
- Eventually settles into “pattern” (eigenfunction)
Local theory:

- Based on distribution of Lyapunov exponents.
Local vs Global Regimes of Mixing

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  - [Antonsen et al., Phys. Fluids (1996)]
  - [Balkovsky and Fouxon, PRE (1999)]
  - [Son, PRE (1999)]

Global theory:

- Eigenfunction of advection–diffusion operator.

Today: Try to connect the two pictures.

Cannot often do this! Map allows (mostly) analytical results.
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  • [Son, PRE (1999)]  Statistical model

Global theory:

• Eigenfunction of advection–diffusion operator.
  
  • [Pierrehumbert, Chaos Sol. Frac. (1994)]  Strange eigenmode
  • [Fereday et al., Wonhas and Vassilicos, PRE (2002)]  Baker’s map
  • [Sukhatme and Pierrehumbert, PRE (2002)]
  • [Fereday and Haynes (2003)]  Unified description
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A Bit of History

Eulerian (spatial) coordinates are due to...
Eulerian (spatial) coordinates are due to...

d’Alembert
A Bit of History

... and Lagrangian (material) coordinates to...

d’Alembert  
Euler
The people responsible for the confusion...
The people responsible for the confusion...

Lagrange

Dirichlet

(See footnote in Truesdell, *The Kinematics of Vorticity.*)
We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0, 1]^2$,

$$\mathcal{M}(x) = \mathbb{M} \cdot x + \phi(x),$$

where

$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad \phi(x) = \frac{\varepsilon}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix};$$

$\mathbb{M} \cdot x$ is the Arnold cat map.

The map $\mathcal{M}$ is area-preserving and chaotic.

For $\varepsilon = 0$ the stretching of fluid elements is homogeneous in space.

For small $\varepsilon$ the system is still uniformly hyperbolic.
Advection and Diffusion: Eulerian Viewpoint

Iterate the map and apply the **heat operator** to a scalar field (which we call **temperature** for concreteness) distribution $\theta^{(i-1)}(x)$,

$$\theta^{(i)}(x) = H_\kappa \theta^{(i-1)}(M^{-1}(x))$$

where $\kappa$ is the **diffusivity**, with the **heat operator** $H_\kappa$ and kernel $h_\kappa$

$$H_\kappa \theta(x) := \int_{T^2} h_\kappa(x - y) \theta(y) \, dy;$$

$$h_\kappa(x) = \sum_k \exp(2\pi i k \cdot x - k^2 \kappa).$$

In other words: **ad vect** instantaneously and then **diffuse** for one unit of time.
Fourier expand $\theta^{(i)}(x)$,

$$\theta^{(i)}(x) = \sum_k \hat{\theta}_k^{(i)} e^{2\pi i k \cdot x}.$$  

The effect of advection and diffusion becomes

$$\hat{\theta}_k^{(i)}(x) = \sum_q \mathbb{T}_{kq} \hat{\theta}_q^{(i-1)},$$  

with the transfer matrix,

$$\mathbb{T}_{kq} := \int_{\mathbb{T}^2} \exp \left( 2\pi i (q \cdot x - k \cdot \mathcal{M}(x)) - \kappa q^2 \right) \, dx,$$

$$= e^{-\kappa q^2} \delta_{0,Q_2} i^{Q_1} J_{Q_1} \left( (k_1 + k_2) \varepsilon \right), \quad Q := k \cdot \mathcal{M} - q,$$

where the $J_Q$ are the Bessel functions of the first kind.
In the absence of diffusion ($\kappa = 0$) the variance $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{\mathbb{T}^2} |\theta^{(i)}(\bm{x})|^2 \, d\bm{x} = \sum_k \sigma_k^{(i)}, \quad \sigma_k^{(i)} := |\hat{\theta}_k^{(i)}|^2$$

is preserved. (We assume the spatial mean of $\theta$ is zero.) For $\kappa > 0$ the variance decays.

We consider the case $\kappa \ll 1$, of greatest practical interest.
Variance: A Measure of Mixing

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Three phases:

- The variance is initially constant;
- It then undergoes a rapid superexponential decay;
- $\theta^{(i)}$ settles into an eigenfunction of the A–D operator that sets the exponential decay rate.
Variance: 5 iterations for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$
Eigenfunction for $\varepsilon = 0.3$ and $\kappa = 10^{-3}$

(Renormalised by decay rate)

$i = 25$

$i = 30$
Decay Rate

For small $\varepsilon$, the dominant Bessel function is $J_1$, so the decay factor $\mu^2$ for the variance is given by

$$\mu = \left| T_{(0,1),(0,1)} \right| = e^{-\kappa} J_1(\varepsilon) = \frac{1}{2} \varepsilon + O(\kappa \varepsilon, \varepsilon^2).$$

Hence, for small $\varepsilon$ the decay rate is limited by the $(0, 1)$ mode. The decay rate is independent of $\kappa$ for $\kappa \rightarrow 0$. 
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This is an analogous result to the baker’s map [Fereday et al., Wonhas and Vassilicos, PRE (2002)]. Here the agreement with numerical results is good for \( \varepsilon \) quite close to unity.
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In the baker’s map the discontinuity implies a slow convergence of the Fourier modes. However, it is a one-dimensional problem.
Decay Rate as $\kappa \to 0$
Lagrangian Viewpoint

- Puzzle: Superexponential decay in Lagrangian coordinates.
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Discover what large-scale eigenfunction looks like in Lagrangian coordinates (hint: they are not eigenfunctions!).
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• Why do this? The two viewpoints are a priori unrelated, because they for these highly-chaotic systems they are connected by an extremely convoluted (i.e., inaccessible) transformation!
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- Why do this? The two viewpoints are a priori unrelated, because they for these highly-chaotic systems they are connected by an extremely convoluted (i.e., inaccessible) transformation!
- But must give same answer for a scalar quantity like the decay rate.
Advection and Diffusion: Eulerian to Lagrangian

Advection-diffusion (A–D) equation:

\[
\partial_t \theta + \mathbf{v} \cdot \partial_x \theta = \tilde{\kappa} \partial_x^2 \theta.
\]
Advection-diffusion (A–D) equation:

\[ \partial_t \theta + \mathbf{v} \cdot \partial_x \theta = \kappa \partial^2_x \theta. \]

We define Lagrangian coordinates \( \mathbf{X} \) by

\[ \dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{X}. \]
Advection and Diffusion: Eulerian to Lagrangian

Advection-diffusion (A–D) equation:
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We define Lagrangian coordinates \( X \) by
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Transform A–D equation to Lagrangian coordinates,
\[ \dot{\theta} = \partial_X (\mathbb{D} \cdot \partial_X \theta). \]

Anisotropic diffusion tensor, in terms of metric or Cauchy–Green strain tensor:
\[ \mathbb{D} := \kappa \, g^{-1}; \quad g_{pq} := \sum_i \frac{\partial x^i}{\partial X^p} \frac{\partial x^i}{\partial X^q}. \]
Velocity field doesn’t enter the Lagrangian equation directly: regard the time dependence in $\mathbb{D}$ as given by map rather than flow.

The solution of the A–D equation in Fourier space is then

$$\hat{\theta}_k^{(i)} = \sum_{\ell} \exp(\mathcal{G}_k^{(i)}) \hat{\theta}_\ell^{(i-1)},$$

where $i$ denotes the $i$th iterate of the map, and

$$\mathcal{G}_k^{(i)} = -4\pi^2 T \int_{\mathbb{T}^2} (\mathbf{k} \cdot \mathbb{D}^{(i)} \cdot \mathbf{\ell}) e^{-2\pi i (\mathbf{k} - \mathbf{\ell}) \cdot \mathbf{X}} \, d^2 \mathbf{X}.$$
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$$\hat{\theta}^{(i)}_k = \sum_{\ell} \exp(\mathcal{G}^{(i)})_{k\ell} \hat{\theta}^{(i-1)}_\ell,$$

where $i$ denotes the $i$th iterate of the map, and

$$\mathcal{G}^{(i)}_{k\ell} = -4\pi^2 T \int_{\mathbb{T}^2} (k \cdot \mathbb{D}^{(i)} \cdot \ell) e^{-2\pi i (k-\ell) \cdot X} \, d^2 X.$$

This is an exact result, but the great difficulty lies in calculating the exponential of $\mathcal{G}^{(i)}$. We shall accomplish this perturbatively.
\[ \mathcal{M}(\mathbf{x}) = \mathbf{M} \cdot \mathbf{x} + \phi(\mathbf{x}), \]

\[ \mathbf{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}; \quad \phi(\mathbf{x}) = \frac{\varepsilon}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix}; \]

The eigenvalues of \( \mathbf{M} \) are

\[ \Lambda_u = \Lambda = \frac{1}{2}(3 + \sqrt{5}) = \cot^2 \theta, \quad \Lambda_s = \Lambda^{-1} = \frac{1}{2}(3 - \sqrt{5}) = \tan^2 \theta \]

and the corresponding eigenvectors,

\[ (\mathbf{u} \; \mathbf{s}) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \]
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\[
\Lambda^{-1} \quad \Lambda \\
\theta
\]
The coefficients of expansion and characteristic directions for the linear cat map are uniform in space. Perturb off this.

To leading order in $\varepsilon$, the coefficient of expansion is written as

$$
\Lambda_{\varepsilon}^{(i)} = \Lambda^i \left( 1 + \varepsilon \eta^{(i)} \right)
$$

where $\Lambda$ is the coefficient of expansion for the unperturbed cat map; the perturbed eigenvectors are similarly written

$$
\hat{u}_{\varepsilon}^{(i)} = \hat{u} + \varepsilon \zeta^{(i)} \hat{s}, \quad \hat{s}_{\varepsilon}^{(i)} = \hat{s} - \varepsilon \zeta^{(i)} \hat{u}.
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Simple application of matrix perturbation theory to Jacobian matrix of the map. The symmetrised Jacobian is the metric:

$$g_{\varepsilon}^{(i)} = [\Lambda_{\varepsilon}^{(i)}]^2 \hat{u}_{\varepsilon}^{(i)} \hat{u}_{\varepsilon}^{(i)} + [\Lambda_{\varepsilon}^{(i)}]^{-2} \hat{s}_{\varepsilon}^{(i)} \hat{s}_{\varepsilon}^{(i)}.$$
Perturbation Results

\[ \Lambda^{(i)}_{\varepsilon} = \Lambda^i (1 + \varepsilon \eta^{(i)}), \quad \hat{u}^{(i)}_{\varepsilon} = \hat{u} + \varepsilon \zeta^{(i)} \hat{s}, \]

\[ \eta^{(i)} = \frac{1}{2} \sin 2\theta \sum_{j=0}^{i-1} \cos \left( 2\pi (M^j \cdot X)_1 \right); \]

\[ \zeta^{(i)} = \frac{1}{\Lambda^{2i} - \Lambda^{-2i}} (\zeta_+^{(i)} + \zeta_-^{(i)}), \]

\[ \zeta_{\pm}^{(i)} = \frac{1}{2} (\cos 2\theta \mp 1) \sum_{j=0}^{i-1} \Lambda^{\pm 2(i-j)} \cos \left( 2\pi (M^j \cdot X)_1 \right). \]

Observe that the perturbation to the eigenvectors converges exponentially, as required.
Perturbed Metric Tensor

\[ \mathbb{D}^{(i)} = \kappa \left[ g^{(i)}_\varepsilon \right]^{-1}; \quad \left[ g^{(i)}_\varepsilon \right]^{-1} = \left[ \Lambda^{(i)}_\varepsilon \right]^2 \hat{s}^{(i)}_\varepsilon \hat{s}^{(i)}_\varepsilon + \left[ \Lambda^{(i)}_\varepsilon \right]^{-2} \hat{u}^{(i)}_\varepsilon \hat{u}^{(i)}_\varepsilon. \]

To leading order in \( \varepsilon \), we have

\[ \left[ g^{(i)}_\varepsilon \right]^{-1} = \Lambda^{2i} \hat{s} \hat{s} + \Lambda^{-2i} \hat{u} \hat{u} + 2\varepsilon \eta^{(i)} \left( \Lambda^{2i} \hat{s} \hat{s} - \Lambda^{-2i} \hat{u} \hat{u} \right) \]

\[ - \varepsilon \zeta^{(i)} \left( \Lambda^{2i} - \Lambda^{-2i} \right) \left( \hat{u} \hat{s} + \hat{s} \hat{u} \right), \]

where the only functions of \( X \) are \( \eta^{(i)} \) and \( \zeta^{(i)} \).
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- \varepsilon \zeta^{(i)} (\Lambda^{2i} - \Lambda^{-2i}) (\hat{u} \hat{s} + \hat{s} \hat{u}),
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where the only functions of \( X \) are \( \eta^{(i)} \) and \( \zeta^{(i)} \).

Recall the solution to the A–D equation:

\[
\hat{\theta}^{(i)}_k = \sum_{\ell} \exp(G^{(i)}_{k\ell}) \hat{\theta}^{(i-1)}_{\ell}.
\]
The Exponent $G^{(i)}$

\[ G^{(i)}_{k\ell} = -4\pi^2 T \int_{\mathbb{T}^2} (k \cdot D^{(i)} \cdot \ell) e^{-2\pi i (k-\ell) \cdot X} \, d^2 X \]

\[ = A^{(i)}_{k\ell} + \varepsilon B^{(i)}_{k\ell} \]
\[ G_{k \ell}^{(i)} = -4 \pi^2 T \int_{T^2} (\mathbf{k} \cdot \mathbf{D}^{(i)} \cdot \mathbf{\ell}) e^{-2 \pi i (\mathbf{k} - \mathbf{\ell}) \cdot \mathbf{X}} \, d^2 \mathbf{X} \]

\[ = A_{k \ell}^{(i)} + \varepsilon B_{k \ell}^{(i)} \]

where

\[ A_{k \ell}^{(i)} = -\kappa \left( \Lambda^{2i} k_s^2 + \Lambda^{-2i} k_u^2 \right) \delta_{k \ell}, \quad \kappa := 4\pi^2 \tilde{\kappa} T \]

\[ B_{k \ell}^{(i)} = -\kappa \left( 2 \left( \Lambda^{2i} k_s \ell_s - \Lambda^{-2i} k_u \ell_u \right) \eta_{k \ell}^{(i)} \right. \]

\[ - (k_u \ell_s + k_s \ell_u) \left( \zeta_+^{(i)} k \ell + \zeta_-^{(i)} k \ell \right). \]

with \( k_u := (\mathbf{k} \cdot \mathbf{\hat{u}}), \) \( k_s := (\mathbf{k} \cdot \mathbf{\hat{s}}). \)
The diagonal part, \( A^{(i)} \), inexorably leads to superexponential decay of variance, because it grows exponentially. Upon making use of the Fourier-transformed \( \zeta^{(i)} \) and \( \eta^{(i)} \), we find

\[
B_{k\ell}^{(i)} = -\frac{1}{2} \kappa \sum_{j=0}^{i-1} B_{k\ell}^{ij} \left( \delta_{k,\ell+\hat{e}_1\cdot\mathbf{M}^j} + \delta_{k,\ell-\hat{e}_1\cdot\mathbf{M}^j} \right)
\]

\[
B_{k\ell}^{ij} = \sin 2\theta \left( \Lambda^{2i} k_s \ell_s - \Lambda^{-2i} k_u \ell_u \right)
+ (k_u \ell_s + k_s \ell_u) \left( \Lambda^{2(i-j)} \sin^2 \theta - \Lambda^{-2(i-j)} \cos^2 \theta \right).
\]

So \( B^{(i)} \) is not diagonal (it couples different modes to each other).

\[\Rightarrow\] Dispersive in Fourier space.
But can we Compute the Exponential, $\exp(\mathcal{G}^{(i)})$?

To leading order in $\varepsilon$, for $A$ diagonal, we have $\mathcal{G}^{(i)} = A^{(i)} + \varepsilon B^{(i)}$,

$$[\exp(A^{(i)} + \varepsilon B^{(i)})]_{k\ell} = e^{A_{k\ell}^{(i)}} \delta_{k\ell} + \varepsilon E_{k\ell}^{(i)}; \quad E_{k\ell}^{(i)} = B_{k\ell}^{(i)} \frac{e^{A_{kk}^{(i)}} - e^{A_{\ell\ell}^{(i)}}}{A_{kk}^{(i)} - A_{\ell\ell}^{(i)}}.$$

- From Eulerian considerations, we know we must avoid superexponential decay of $\theta^{(i)}$ for long times.
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- However, the \( \Lambda^{2i} \) term in \( A_{kk}^{(i)} \) precludes any optimism about the situation: it dooms us to a grim superexponential death.
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- From Eulerian considerations, we know we must avoid superexponential decay of \( \theta^{(i)} \) for long times.
- However, the \( \Lambda^{2i} \) term in \( A^{(i)}_{kk} \) precludes any optimism about the situation: it dooms us to a grim superexponential death.
- For \( \varepsilon = 0 \), this is indeed what happens. But for a finite value of \( \varepsilon \), the \( E \) term breaks the diagonality of \( G \), so that given some initial set of wavevectors, the variance contained in those modes can be transferred elsewhere.
• Impractical to take the matrix exponential for large matrices.

As \( i \) increases, most modes are damped as \( \exp(-2i\mathbf{k}_s^2 + 2i\mathbf{k}_u^2) \), except for those that have very small \( \mathbf{k}_s \), i.e., those that are aligned with \( \mathbf{u} \).

Just let computer take care of pruning via underflow!

The surviving modes need to become more and more aligned with \( \mathbf{u} \) as time goes on.
- Impractical to take the matrix exponential for large matrices.
- Perturbative expansion sidesteps this problem.

However, still need to go to extremely high wavenumber... impossible to use mesh, since would have to refine exponentially fast.

So keep track of only the required wavevectors: their number should grow exponentially... but it doesn't!

This is because as $i$ increases, most modes are damped as $\exp(-2i k_s^2 s + 2i k_s^2 u)$, except for those that have very small $k_s = (k_s^s)^i$, i.e., those that are aligned with $^s u$.

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The surviving modes need to become more and more aligned with $^s u$ as time goes on.
A Few Words about Numerics

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\[ \text{as } n \to \infty, \text{modes are damped as } e^{2\pi i k^2 s + 2\pi i k^2 u}, \text{except for those that have very small } k^2 s = (k^2)^s, \text{ i.e. those that are aligned with } \hat{u}. \]

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A Few Words about Numerics

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• The surviving modes need to become more and more aligned with $\hat{u}$ as time goes on.
Comparison: Eulerian and Lagrangian Views

\[ \varepsilon = 10^{-4} \]

\[
\begin{align*}
\kappa = 0.1 \\
\kappa = 0.01 \\
\kappa = 0.001 \\
\kappa = 0.0001 \\
\text{Lagrangian}
\end{align*}
\]
Iteration $= 4$

$\kappa = 0.01$

$|\text{Eulerian} - \text{Lagrangian}|$ vs $\varepsilon$

The graph shows the convergence of the Eulerian and Lagrangian methods as $\varepsilon$ decreases. The convergence rate is approximately $\varepsilon^{-2}$. The notation $\text{Eulerian} - \text{Lagrangian}$ likely refers to the difference between the Eulerian and Lagrangian solutions, which is a common measure of accuracy in fluid dynamics studies.
Rescaled Pattern for $i = 6, \ldots, 12$

$\frac{\log_{10} \text{amplitude (rescaled)}}{\varepsilon = 10^{-4}}$

$k$ (aligned with $\hat{u}$, scaled by $\Lambda^i$)

$\kappa = 0.1$
Conclusions

• In the Eulerian view, large-scale eigenmode dominates exponential phase, as for baker’s map.
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Still some kinks to iron out!