Topological detection of Lagrangian coherent structures

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Sparse trajectories and material loops

How do we efficiently detect trajectories that ‘bunch’ together?

[movie 1]
Growth of loops enclosing trajectories

For 3 trajectories, look at the growth of curves:

We use the braid generator notation: $\sigma_i$ means the clockwise interchange of the $i$th and $(i + 1)$th trajectory. (Inverses are counterclockwise.)

The motion above is denoted $\sigma_1 \sigma_2^{-1}$. 
The rate of growth $h = \log \lambda$ is called the topological entropy.

But how do we find the rate of growth of curves for motions on the disk?

For 3 trajectories it’s easy: the entropy for $\sigma_1 \sigma_2^{-1}$ is $h = \log \varphi^2$, where $\varphi$ is the Golden Ratio!

For more trajectories, use Moussafir iterative technique (2006).

Iterating a loop

It is well-known that the entropy can be obtained by applying the trajectories to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;

2. Need a simple way of transforming the loop according to the trajectories.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.
Solution to problem 1: Loop coordinates

What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the crossing numbers count intersections with vertical lines:
Dynnikov coordinates

Now take the difference of crossing numbers:

\[ a_i = \frac{1}{2} (\mu_{2i} - \mu_{2i-1}) , \]
\[ b_i = \frac{1}{2} (\nu_i - \nu_{i+1}) \]

for \( i = 1, \ldots, n - 2 \).

The vector of length \( 2n - 4 \),

\[ \mathbf{u} = (a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2}) \]

is called the Dynnikov coordinates of a loop.

There is a one-to-one correspondence between closed loops and these coordinates: you can’t do it with fewer than \( 2n - 4 \) numbers.
Solution to problem 2: Action on coordinates

Moving the points according to a braid generator changes some crossing numbers:

There is an explicit formula for the change in the coordinates!
Action on loop coordinates

The update rules for $\sigma_i$ acting on a loop with coordinates $(a, b)$ can be written

\[
\begin{align*}
a'_{i-1} &= a_{i-1} - b^+_{i-1} - (b^+_i + c_{i-1})^+ , \\
b'_{i-1} &= b_i + c^-_{i-1} , \\
a'_i &= a_i - b^-_i - (b^-_{i-1} - c_{i-1})^- , \\
b'_i &= b_{i-1} - c^-_{i-1} ,
\end{align*}
\]

where

\[
f^+ := \max(f, 0), \quad f^- := \min(f, 0).
\]

\[
c_{i-1} := a_{i-1} - a_i - b^+_i + b^-_{i-1}.
\]

This is called a piecewise-linear action.

Easy to code up (see for example Thiffeault (2010)).
For a specific set of trajectories, say as given by the braid \( \sigma_3^{-1} \sigma_2^{-1} \sigma_3^{-1} \sigma_2 \sigma_1 \), we can easily see the exponential growth of \( L \) and thus measure the entropy:
$m$ is the number of times the braid acted on the initial loop.
Lagrangian Coherent Structures

- There is a lot more information in the braid than just entropy;
- For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
- Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
- For now: regions are not ‘leaky.’
Sample system: Modified Duffing oscillator

\[ \dot{x} = y + \alpha \cos \omega t, \]
\[ \dot{y} = x(1 - x^2) + \gamma \cos \omega t - \delta y, \]

+ rotation to further hide two regions. \( \alpha = .1, \gamma = .14, \delta = .08, \omega = 1. \)
Growth of loops

Coding of loops

LCS

Conclusions

References

Growth of a vast number of loops

Left: semilog plot; Right: linear plot of slow-growing loops.

Clearly two types of loops!
What do the slowest-growing loops look like?

[(c) appears because the coordinates also encode ‘multiloops.’]
Computational complexity

Here’s the bad news:

• There are an infinite number of loops to consider.
• But we don’t really expect hyper-convoluted initial loops (nor do we care so much about those).
• Even if we limit ourselves to loops with Dynnikov coordinates between $-1$ and $1$, this is still $3^{2n-4}$ loops.
• This is too many... can only treat about 10–11 trajectories using this direct method.
An improved method: Pair-loops

The biggest problem is that we only look at whether a loop grows or not. But there is a lot more information to be found in how a loop entangles the trajectories as it evolves.

Consider loops that enclose only two trajectories at once. More involved analysis, but scales much better with $n$. 
Improvement

Run times in seconds:

<table>
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<tr>
<th># of trajectories</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>20</th>
</tr>
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<td>direct method</td>
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<td>53</td>
<td>462</td>
<td>3445</td>
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<tr>
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<td>12.3</td>
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<td>20</td>
<td>128</td>
</tr>
</tbody>
</table>

Bottleneck for the pair-loop method is finding the non-growing loops. (Should scale as $n^2$ for large enough $n$.)

The downside is that the pair-loop method is much more complicated. But in the end it accomplishes the same thing.
A physical example: Rod stirring device
Conclusions

• Having trajectories undergo ‘braiding’ motion guarantees a minimal amount of entropy (stretching of material lines);
• This idea can also be used on fluid particles to estimate entropy;
• Need a way to compute entropy fast: loop coordinates;
• There is a lot more information in this braid: extract it! (Lagrangian coherent structures);
• Is this useful? We need good physical problems to try it on!
• See Thiffeault (2005, 2010) and soon preprint by Allshouse & Thiffeault.
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References


