Lyapunov Exponents and Transport in 2D Flows

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What is the Deal

We are interested in the advection-diffusion equation:

$$\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = \frac{1}{\rho} \nabla \cdot (\rho D \nabla \phi)$$

where the Eulerian velocity field $\mathbf{v}(\mathbf{x}, t)$ is some prescribed time-dependent flow, which may or may not be chaotic. The quantity $\phi$ represents the concentration of some passive scalar, $\rho$ is the density, and $D$ is the diffusion coefficient.

We assume that the Lagrangian dynamics are strongly chaotic ($\lambda L^2/D \gg 1$).
Lagrangian Coordinates

The trajectory of a fluid element in Eulerian coordinates $x$ satisfies

$$\frac{dx}{dt} (\xi, t) = v(x(\xi, t), t),$$

where $\xi$ are Lagrangian coordinates which label fluid elements. The usual choice is to take as initial condition $x(\xi, t = 0) = \xi$, which says that fluid elements are labeled by their initial position.

$x = x(\xi, t)$ is thus the transformation from Lagrangian ($\xi$) to Eulerian ($x$) coordinates.

This transformation gets horrendously complicated as time evolves.
Lyapunov Exponents

The rate of exponential separation of neighbouring Lagrangian trajectories is measured by Lyapunov exponents

\[ \lambda_\infty = \lim_{t \to \infty} \frac{1}{t} \ln \| (T_x v) w_0 \|, \]

where \( T_x v \) is the tangent map of the velocity field (the matrix \( \frac{\partial v}{\partial x} \)) and \( w_0 \) is some constant vector.

Lyapunov exponents converge very slowly. So, for practical purposes we are always dealing with finite-time Lyapunov exponents.
(Welander, 1955)
The Idea

- Can we characterize the **spatial** and **temporal** evolution of finite-time Lyapunov exponents in a generic manner?

- Can we quantify the impact of these exponents on diffusion? Tang and Boozer brought the fancy tools of differential geometry to bear on this problem.

- **Results**: a generic functional form for the time evolution of finite-time Lyapunov exponents, and a relation between their spatial dependence and the shape of the stable manifolds.
A little differential geometry …

The Jacobian of the transformation from Lagrangian ($\xi$) to Eulerian ($\xi$) coordinates

$$J^i_j \equiv \frac{\partial x^i}{\partial \xi^j}$$

The Jacobian tells us how tensors transform:

- **Covariant:**
  $$\tilde{V}_j = J^k_j V_k,$$

- **Contravariant:**
  $$\tilde{W}^i = J^i_k W^k.$$
Measuring distances

The distance between two infinitesimally separated points in Eulerian space is given by

\[ ds^2 = d\mathbf{x} \cdot d\mathbf{x} = \delta_{ij} \, dx^i \, dx^j . \]

Therefore, in Lagrangian coordinates distances are given by

\[ ds^2 = \delta_{ij} \left( \frac{dx^i}{d\xi^k} \, d\xi^k \right) \left( \frac{dx^j}{d\xi^\ell} \, d\xi^\ell \right) = (J^i_k \, \delta_{ij} \, J^j_\ell) \, d\xi^k \, d\xi^\ell . \]
The Metric Tensor

The tensor $\delta_{ij}$ is a *metric* in the Eulerian (Euclidean) space. The tensor

$$g_{k\ell}(\xi, t) \equiv \sum_i J_i^k J_i^{\ell} = (J^T J)_{k\ell}$$

is the same metric tensor but in the Lagrangian coordinate system.

Since the metric tells us about the distance between two neighbouring Lagrangian trajectories, its eigenvalues are related to the finite-time Lyapunov exponents.
2-D Incompressible Flow

We will now restrict ourselves to a 2-D, incompressible velocity field \( \mathbf{v} \). This means that

\[
\det g = (\det J)^2 = 1.
\]

Now, \( g \) is a positive-definite symmetric matrix, which implies that it has real positive eigenvalues, \( \Lambda(\xi, t) \geq 1 \) and \( \Lambda^{-1}(\xi, t) \leq 1 \), and orthonormal eigenvectors \( \hat{e}(\xi, t) \) and \( \hat{s}(\xi, t) \):

\[
g_{k\ell}(\xi, t) = \Lambda e_k e_\ell + \Lambda^{-1} s_k s_\ell
\]

The finite-time Lyapunov exponents are given by

\[
\lambda(\xi, t) = \ln \Lambda(\xi, t)/2t
\]
Stable and Unstable Directions

At a fixed coordinate $\xi$:

The stable and unstable manifolds $\hat{e}(\xi, t)$ and $\hat{s}(\xi, t)$ converge exponentially to their asymptotic values $\hat{e}_\infty(\xi)$ and $\hat{s}_\infty(\xi)$, whereas Lyapunov exponents converge logarithmically.
\( \hat{s}_\infty \)-line for the standard map with \( k = 1.5 \).
\( \hat{s}_\infty \)-line for the standard map with \( k = 50 \).
\[ \psi(x, t) = Ak^{-1}(\sin kx \sin \pi y + \epsilon \cos \omega t \cos kx \cos \pi y) \]

Oscillating convection rolls \((A = k = \epsilon = \omega = 1)\).
The Advection-diffusion Equation

Under the coordinate change to Lagrangian variables the diffusion term becomes

\[ \nabla \cdot (D \nabla \phi) = \frac{\partial}{\partial x^i} (D \delta^{ij} \frac{\partial \phi}{\partial x^j}) = \frac{\partial}{\partial \xi^i} (D g^{ij} \frac{\partial \phi}{\partial \xi^j}). \]

In Lagrangian coordinates the diffusivity becomes \( D g^{ij} \): it is no longer isotropic.

The advection-diffusion equation is thus just the diffusion equation,

\[ \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial \xi^i} (D g^{ij} \frac{\partial \phi}{\partial \xi^j}), \]

because by construction the advection term drops out.
Diffusion along $\hat{s}_\infty$ and $\hat{e}_\infty$

The diffusion coefficients along the $\hat{s}_\infty$ and $\hat{e}_\infty$ lines are

$$D^{ss} = s_\infty^i (Dg^{ij}) s_\infty^j = D \exp(2\lambda t),$$

$$D^{ee} = e_\infty^i (Dg^{ij}) e_\infty^j = D \exp(-2\lambda t).$$

We see that $D^{ee}$ goes to zero exponentially quickly. Hence, essentially all the diffusion occurs along the $\hat{s}_\infty$ line.
Differential geometry tells us if a metric describes a flat space, then its Riemann curvature tensor must vanish in every coordinate system.

After some tedious algebra, we find this implies that the quantity

\[ \hat{s}_\infty \cdot \nabla_0 \lambda(\xi, t) t + \nabla_0 \cdot \hat{s}_\infty \]

converges to 0 exponentially. Hence, it can be shown that the finite-time Lyapunov exponents must have the form

\[ \lambda(\xi, t) = \frac{\tilde{\lambda}(\xi)}{t} + \frac{f(\xi, t)}{\sqrt{t}} + \lambda_\infty, \]

where \( \hat{s}_\infty \cdot \nabla_0 f = 0 \) (the \( 1/\sqrt{t} \) factor comes from known results on the variance of the exponents).
Example:

Dotted: Numerical

Solid: $0.305/t + 0.175/\sqrt{t} + 0.117$

Allows us to determine $\lambda_\infty = 0.117$ rapidly and accurately.
Standard map, 5th iteration, $k = 50$ (curvature $\kappa \equiv (\hat{s}_\infty \cdot \nabla_0)\hat{s}_\infty$).
Standard map, 5th iteration, $k = 50$. 
Conclusions

- Diffusion occurs overwhelmingly along the stable direction.
- The spatial dependence of Lyapunov exponents along $\hat{s}$ lines is contained in the smooth function $\tilde{\lambda}(\xi)$, which decays as $1/t$.
- The notoriously slow convergence of Lyapunov exponents is embodied in the function $f(\xi, t)$, which is constant on $\hat{s}$ lines and decays as $1/\sqrt{t}$.
- Relation between $\hat{s}_\infty(\xi)$, $\kappa \equiv (\hat{s}_\infty \cdot \nabla_0)\hat{s}_\infty$, and $\tilde{\lambda}(\xi)$.
- Sharp bends in the $\hat{s}$ line lead to locally small finite-time Lyapunov exponents (diffusion is hindered).
- Test on flows.