Braids and Dynamics
Frontiers in Theory and Modelling with Scarce Data

Jean-Luc Thiffeault

Department of Mathematics
University of Wisconsin – Madison

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Taffy is a type of candy.

Needs to be pulled: this aerates it and makes it lighter and chewier.

We can assign a growth: length multiplier per period.

(Here \((1 + \sqrt{2})^2 \ldots \) more later.)

[movie by M. D. Finn]
making candy cane

[Wired: This Is How You Craft 16,000 Candy Canes in a Day]
four-pronged taffy puller

http://www.youtube.com/watch?v=Y7t1HDsquVM

[MacKay (2001); Halbert & Yorke (2014)]
Experimental device for kneading bread dough:

[Department of Food Science, University of Wisconsin. Photos by J-LT.]
the mixograph as a braid

Encode the topological information as a sequence of generators of the Artin braid group $B_n$.

Equivalent to the 7-braid

$$\sigma_3\sigma_2\sigma_3\sigma_5\sigma_6^{-1}\sigma_2\sigma_3\sigma_4\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_5$$

The growth is the largest root of

$$x^8 - 4x^7 - x^6 + 4x^4 - x^2 - 4x + 1 \approx 4.186$$

Compare to taffy pullers: 5.828
braids and rod-stirring

4 + 1 rods

There is an underlying *topological description* of the rod motions.

Regard fluid motion as a *map from the domain to itself*.

Map can be classified into *finite-order*, *reducible*, and *pseudo-Anosov*.

Pseudo-Anosov maps have some inherent *topological mixing*, which is a kind of chaotic behavior.

Characterized by *topological entropy*, related to Lyapunov exponents.
the topological program

- Consider a motion of stirring elements, such as rods.
- Determine if the motion is isotopic to a pseudo-Anosov mapping.
- Compute topological quantities, such as foliation, entropy, etc.
- Analyze and optimize.
insight: do we need the rods?

ghost rods (‘tiges fantômes’)

Topological analysis can be done on other objects than rods – for instance, islands or unstable periodic orbits.

We simply follow the islands and examine the braid they form, which gives us bounds on topological entropy.

In this framework we call the islands ghost rods.


[implemented by Stremler & Chen (2007); Thiffeault et al. (2009); Binder (2010); Stremler et al. (2011)]
One of the best examples of ghost rods is from Stremler et al. (2011):

The islands are made to follow the $\sigma_2 \sigma_1^{-1}$ stirring protocol by clever wall motions! (viscous Stokes flow)

oceanic float trajectories
What can we measure?

- single-particle dispersion (not a good use of all data)
- correlation functions (useful)
- Lyapunov exponents (some luck needed!)

Another possibility:

Compute the braid group generators $\sigma_i$ for the float trajectories (convert to a sequence of symbols), then look at how loops grow. Obtain a topological entropy for the motion (similar to Lyapunov exponent, or to the ‘growth’ of taffy pullers).
It is well-known that the entropy can be obtained by applying the motion of the punctures to a closed curve (loop) repeatedly, and measuring the growth of the length of the loop (Bowen, 1978).

The problem is twofold:

1. Need to keep track of the loop, since its length is growing exponentially;
2. Need a simple way of transforming the loop according to the motion of the punctures.

However, simple closed curves are easy objects to manipulate in 2D. Since they cannot self-intersect, we can describe them topologically with very few numbers.
What saves us is that a closed loop can be uniquely reconstructed from the number of intersections with a set of curves. For instance, the **Dynnikov coordinates** involve intersections with vertical lines:
Moving the punctures according to a braid generator changes some crossing numbers:

There is an explicit formula for the change in the coordinates! [Dynnikov (2002); Moussafir (2006); Hall & Yurttaş (2009); Thiffeault (2010)]
For a specific rod motion, say as given by the braid $\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_1$, we can easily see the exponential growth of $L$ and thus measure the entropy:
growth of $L$ (2)

$m$ is the number of times the braid acted on the initial loop.

10 floats from Davis’ Labrador sea data:

Floats have an entanglement time of about 50 days — timescale for horizontal stirring.

Source: WOCE subsurface float data assembly center (2004)
• There is a lot more information in the braid than just entropy;
• For instance: imagine there is an isolated region in the flow that does not interact with the rest, bounded by Lagrangian coherent structures (LCS);
• Identify LCS and invariant regions from particle trajectory data by searching for curves that grow slowly or not at all.
• [see Haller, G. & Beron-Vera, F. J. (2012). *Physica D*, 241 (20), 1680–1702.]
• **Topological approach:** [Allshouse & Thiffeault (2012); Filippi et al. (2020); Yeung et al. (2020)].
double-gyre coherent structures

braids in the heart

braids in the heart (cont’d)

\[
\begin{align*}
\text{Wr} & \quad \text{writhe of braid} \\
\text{ROA} & \quad \text{Regurgitant Orifice Area} \\
\Gamma^*_d & \quad \text{circulation}
\end{align*}
\]

multiagent modeling: cars at an intersection

braiding in active nematics

complexity of crowd movement

some research directions

• We don’t have solid theory for aperiodic or open braids.
• Computational methods for isotopy class (random entanglements of trajectories – LCS method, see Allshouse & Thiffeault (2012); Filippi et al. (2020); Yeung et al. (2020).
• ‘Designing’ for topological chaos (see Stremler & Chen (2007)).
• Combine with other measures, e.g., mix-norms (Mathew et al., 2005; Lin et al., 2011; Thiffeault, 2012).
references I


