Mixing hits a wall

The mixing pattern is caused by some periodic stirring mechanism.

At every period $T$, some white fluid is shaved from the edge, decreasing the distance $d(t)$.

What happens inside the mixing region? Assume a very simple chaotic mixer, meaning that fluid elements are stretched at rate $\Lambda$ (on average)

The maximum width a dye filament can have is given by the Batchelor length,

$$l = \sqrt{\frac{\kappa}{\Lambda}}$$

$k$ = molecular diffusivity

(Balance between diffusion and strain)
The width of a white filament injected at time $t$ is
\[ \Delta(t) = d(t) - d(t+\tau) \approx -\lambda d(t) \]
since $d$ changes little at each period.

Now, if a white filament is injected at time $\tau < t$, how long does it last? "age" of filament

\[ \Delta(\tau) e^{-\lambda (t-\tau)} = \lambda \]

initial width compression Batchelor length

In other words, solving for $\tau(t)$ in the above gives us the injection time of the oldest filaments still visible at time $t$. Filaments injected before $\tau(t)$ are below the Batchelor length, so they are no longer white.

At some point the injected filament width will equal the Batchelor length:
\[ \Delta(t_B) = \lambda \]

so that $\tau(t_B) = t_B$.

It makes no sense to speak of those filaments as "white," since they are dominated by diffusion. Hence, assume $t < t_B$. 

Now, in the experiments we measure the number of "white pixels" in the central mixing region. This is proportional to the area of white material injected,

\[ A_w(t) \sim d(\tau(t)) - d(t) \]

\[ \Delta(t) = -T \dot{d} = d_0 e^{-\mu t} \]

\[ \Delta(t_B) = \lambda \Rightarrow t_B = \frac{1}{\mu} \log \left( \frac{d_0 \mu T}{\lambda} \right) \]

Solve for \( \tau(t) \):

\[ \Delta_0 e^{-\mu \tau} e^{-\lambda(t-\tau)} = \lambda \]

\[ e^{(\lambda-\mu)\tau} = \left( \frac{\lambda}{\Delta_0} \right) e^{\lambda t} \]

\[ \tau = \frac{1}{\lambda-\mu} \left[ \log \left( \frac{\lambda}{\Delta_0} \right) + \lambda t \right] \]

\[ = \frac{1}{\lambda-\mu} \left( -\mu t_B + \lambda t \right) \]
\[ \tau(t) = \frac{\lambda t - \mu t_B}{\lambda - \mu} \]

\[ \tau - t = \frac{\lambda t - \mu t_B - (\lambda - \mu)t}{\lambda - \mu} = \frac{\mu(t-t_B)}{\lambda - \mu} \]

Now, by assumption: \( \tau < t < t_B \):

\[ t - \tau = \frac{\mu}{\lambda - \mu} (t_B - t) > 0 \]

Hence, for consistency we require \( \mu < \lambda \).

i.e., the rate of approach to the wall is slower than the rate of stretching in the mixing region. Otherwise the wall has no effect: the wall region is depleted faster than filaments can reach the Batchelor scale.

If \( \mu < \lambda \), we have

\[ \Delta_w(t) \sim d(\tau(t)) - d(t) \]

\[ \sim d_o (e^{-\mu\tau} - e^{-\mu t}) \]

\[ \sim d_o e^{-\mu t} (e^{(t-\tau)\mu} - 1) \]

\[ \sim d_o e^{-\mu t} (\exp\left(\frac{\mu^2}{\lambda - \mu}(t_B - t)\right) - 1) \]
We see that for $t \ll t_B$, $A_w(t) \sim e^{-\mu t}$.

So the decay rate of the "white" area is completely dominated by the walls. The central mixing process is efficient ($\lambda > \mu$), but it is "starved" by the boundary.

So what happens in practice?

The "figure-8" protocol shown has a reattachment point.

Let's model it as a steady flow (we can take the time-dependence into account by using a map rather than a flow).

Let $x$ be the "along wall" coordinate.

Taylor in $y$:

$u(x, y) = u_0(x) + u_1(x)y + \ldots$

no slip

$\nabla(x, y) = \nabla_0(x) + \nabla_1(x)y + \nabla_2(x)y^2 + \ldots$

no throughflow

Now, $\nabla \cdot u = 0 \Rightarrow u_1(x)y + \nabla_1(x) + \nabla_2(x)2y = 0$
Equating terms, we find \( N_1(x) = 0, \ N_2(x) = -\frac{1}{2} u'_1(x) \).

Letting \( u_1(x) = A(x) \), get

\[
\begin{align*}
  u(x,y) &= A(x)y + O(y^2) \\
  N(x,y) &= -\frac{1}{2} A'(x)y^2 + O(y^3)
\end{align*}
\]

This is a "boundary layer" form.

A reattachment point corresponds to \( A(x) \) changing sign, say at \( x=0 \), with \( A'(0) > 0 \).

Hence, along the separatrix, we can solve for the motion of a fluid particle,

\[
y' = N(0,y) = -\frac{1}{2} A'(0)y^2
\]

This has solution

\[
y(t) = \frac{y_0}{1 + A'(0)t y_0} \sim \frac{1}{A'(0)t}, \quad t \gg y_0
\]

Note that asymptotically, a particle "forgets" its initial condition \( y_0 \). This explains why material lines "bunch up" against each other faster than they approach the wall.

There is a visible "front".
Now we have our asymptotic form for \( d(t) \):  

\[
d(t) \sim \frac{1}{A'(0) t}, \quad t \gg 1.
\]

Hence, \( \Delta(t) = -T \dot{d}(t) \sim \frac{T}{A'(0) t^2} \).

We have \( A_w(t) = d(t^\ast(t)) - d(t) \) with \( \Delta(t^\ast) e^{-\lambda(t^\ast - t)} = \lambda \).

\[
A_w = \frac{1}{A(0) t^\ast} - \frac{1}{A'(0) t} = \frac{1}{A'(0)} \frac{(t - t^\ast)}{\lambda t}
\]

But also, \( \log \Delta - \lambda (t - t^\ast) = \log \lambda \)

or \( t - t^\ast = \frac{1}{\lambda} \log \left( \frac{\Delta(t)}{\lambda} \right) \)

\( \Delta(t) \) is algebraic, so for large time the RHS is not large. Hence, for large time we must have

\[
t/\tau \approx 1, \quad t \gg 1.
\]

We conclude:

\[
A_w \sim \frac{1}{A'(0)} \frac{\log \left( T/A'(0)t^2 \right)}{\lambda t^2}
\]

\( t \gg 1 \) (but \( t < t_B \))
This power-law decay is consistent with the data.

In a simple "parabolic baker's map" model, reproduces data extremely well for long times.

In experiments, determining $A$ is the most difficult part.

To improve the mixing rate, spin the wall!

$$u(x,y) = U + A(x)y + O(y^2)$$

$$\nu(x,y) = -\frac{1}{2} A'(x)y^2 + O(y^3)$$

The moving wall destroys the reattachment point and replaces it with a hyperbolic fixed point.

Fixed wall: algebraic mixing

Moving wall: exponential

The price to pay is that there is now an unmixed layer near the wall. [There are other means of destroying the reattachment point's.]