Chaotic Mixing and Large-scale Patterns:  
*Mixing in a Simple Map*

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Experiment of Rothstein et al. (1999)

Regular array of magnets

[Rothstein, Henry, and Gollub, Nature 401, 770 (1999)]
Persistent Pattern

Disordered array \((i = 2, 20, 50, 50.5)\)
Evolution of Pattern

- “Striations”
- Smoothed by diffusion
- Eventually settles into “pattern” (eigenfunction)
Local vs Global Regimes of Mixing

Local theory:

• Based on distribution of Lyapunov exponents.
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Average over angles
- Statistical model
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Global theory:

- Eigenfunction of advection–diffusion operator.

Today: Focus on Global theory.
Map allows analytical results (enough said!)
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  • [Pierrehumbert, Chaos Sol. Frac. (1994)]
  • [Fereday et al., Wonhas and Vassilicos, PRE (2002)]
  • [Sukhatme and Pierrehumbert, PRE (2002)]
  • [Fereday and Haynes (2003)]
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• Map allows analytical results (enough said!).
The Map

We consider a diffeomorphism of the 2-torus $\mathbb{T}^2 = [0, 1]^2$,

$$\mathcal{M}(\mathbf{x}) = \mathbb{M} \cdot \mathbf{x} + \phi(\mathbf{x}),$$

where

$$\mathbb{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} ; \quad \phi(\mathbf{x}) = \frac{K}{2\pi} \begin{pmatrix} \sin 2\pi x_1 \\ \sin 2\pi x_1 \end{pmatrix} ;$$

$\mathbb{M} \cdot \mathbf{x}$ is the Arnold cat map.

The map $\mathcal{M}$ is area-preserving and chaotic.

For $K = 0$ the stretching of fluid elements is homogeneous in space.

For small $K$ the system is still uniformly hyperbolic.
Advection and Diffusion

Iterate the map and apply the heat operator to a scalar field (which we call temperature for concreteness) distribution $\theta^{(i-1)}(x)$,

$$\theta^{(i)}(x) = \mathcal{H}_\epsilon \theta^{(i-1)}(\mathcal{M}^{-1}(x))$$

where $\epsilon$ is the diffusivity, with the heat operator $\mathcal{H}_\epsilon$ and kernel $h_\epsilon$

$$\mathcal{H}_\epsilon \theta(x) := \int_{\mathbb{T}^2} h_\epsilon(x - y) \theta(y) \, dy;$$

$$h_\epsilon(x) = \sum_k \exp(2\pi i k \cdot x - k^2 \epsilon).$$

In other words: advect instantaneously and then diffuse for one unit of time.
Transfer Matrix

Fourier expand $\theta^{(i)}(\mathbf{x})$,

$$\theta^{(i)}(\mathbf{x}) = \sum_{k} \hat{\theta}^{(i)}_{k} e^{2\pi i k \cdot \mathbf{x}} .$$

The effect of advection and diffusion becomes

$$\hat{\theta}^{(i)}_{k} = \sum_{q} \mathbb{T}_{kq} \hat{\theta}^{(i-1)}_{q},$$

with the transfer matrix,

$$\mathbb{T}_{kq} := \int_{\mathbb{T}^2} \exp \left( 2\pi i (\mathbf{q} \cdot \mathbf{x} - k \cdot \mathcal{M}(\mathbf{x})) - \epsilon \mathbf{q}^2 \right) \, d\mathbf{x},$$

$$= e^{-\epsilon q^2} \delta_{0,Q_2} i^{Q_1} J_{Q_1} \left( (k_1 + k_2) K \right), \quad Q := k \cdot \mathcal{M} - q,$$

where the $J_{Q}$ are the Bessel functions of the first kind.
In the absence of diffusion ($\epsilon = 0$), the variance $\sigma^{(i)}$ is preserved. (We assume the spatial mean of $\theta$ is zero.) For $\epsilon > 0$ the variance decays. We consider the case $\epsilon \ll 1$, of greatest practical interest.
Variance: A measure of mixing

In the absence of diffusion ($\epsilon = 0$), the variance $\sigma^{(i)}$

$$\sigma^{(i)} := \int_{T^2} |\theta^{(i)}(x)|^2 \, dx = \sum_k \sigma_k^{(i)}, \quad \sigma_k^{(i)} := |\hat{\theta}_k^{(i)}|^2$$

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Three phases:

- The variance is initially constant;
- It then undergoes a rapid superexponential decay;
- $\theta^{(i)}$ settles into an eigenfunction of the A–D operator that sets the exponential decay rate.
Decay of Variance

\[ K = 10^{-3} \]

\[ e^{-15.2i} \]

\[ 10^{-5} \]

\[ 10^{-2} \]

\[ 0.5 \]

\[ 10^2 \]

\[ 10^{-60} \]

\[ 10^{-40} \]

\[ 10^{-20} \]

\[ 10^0 \]

\[ \epsilon = 10 \]

\[ 5 \]

\[ 2 \]

\[ 0 \]

\[ 2 \]

\[ 4 \]

\[ 6 \]

\[ 8 \]

\[ 10 \]

\[ 12 \]

\[ 14 \]

Chaotic Mixing and Large-scale Patterns: – p.10/24
Constant (Stirring) Phase

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- This is the well-known “filamentation” effect in chaotic flows: the stretching and folding action of the flow causes rapid variation of the temperature across the folds.
- Can no longer neglect diffusion after a number of iterations

$$i_1 \approx 1 + (\log \epsilon^{-1} / \log \Lambda^2) \approx 6 \quad \text{for } \epsilon = 10^{-5},$$

where $\Lambda = (3 + \sqrt{5})/2$ is the largest eigenvalue of $M^{-1}$. 
Variance: 5 iterations for $K = 0.3$ and $\epsilon = 10^{-3}$
For small $K$ and $k$, we have $J_0 ((k_1 + k_2)K) \gg J_1 ((k_1 + k_2)K)$. Set $K = 0$ and retain only the $Q_1 = 0$ term in the transfer matrix,

$$ \mathbb{T}_{kq} = e^{-\epsilon q^2} \delta_k, q \cdot M^{-1} + \mathcal{O}((k_1 + k_2)^2 K^2) ; $$
Superexponential Phase

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$$T_{kq} = e^{-\epsilon q^2} \delta_{k, q \cdot \mathbb{M}^{-1}} + \mathcal{O} \left( (k_1 + k_2)^2 K^2 \right);$$

If initially the variance is concentrated in a single wavenumber $q_0$, then after one iteration it will all be in $q_0 \cdot \mathbb{M}^{-1}$, after two in $q_0 \cdot \mathbb{M}^{-2}$, etc.

The length of $q$ is multiplied by the eigenvalue $\Lambda$ at each iteration.
Superexponential Phase

For small $K$ and $k$, we have $J_0 \left( (k_1 + k_2)K \right) \gg J_1 \left( (k_1 + k_2)K \right)$. Set $K = 0$ and retain only the $Q_1 = 0$ term in the transfer matrix,

$$ T_{kq} = e^{-\epsilon q^2} \delta_{k, q \cdot M^{-1}} + O\left((k_1 + k_2)^2 K^2\right); $$

If initially the variance is concentrated in a single wavenumber $q_0$, then after one iteration it will all be in $q_0 \cdot M^{-1}$, after two in $q_0 \cdot M^{-2}$, etc.

The length of $q$ is multiplied by the eigenvalue $\Lambda$ at each iteration. But each at each step the variance is multiplied by the diffusive decay factor $\exp(-\epsilon q^2)$, with $q$ getting exponentially larger.

The net decay is thus superexponential.
Exponential Phase

- In the superexponential phase we described the action as a perfect cascade to large wavenumbers, so that the variance was irrevocably moved to small scales and dissipated extremely rapidly.
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Can the wave term lead to the formation of an eigenfunction of the advection–diffusion operator, which would imply exponential decay?
An Eigenfunction?

Recall:

$$T_{kq} = e^{-\epsilon q^2} \delta_{0,Q_2} i^{Q_1} J_{Q_1} ((k_1 + k_2) K), \quad Q := k \cdot M - q,$$

Consider a matrix element for which $Q_1 \neq 0$. This means that the initial $(q)$ and final $(k)$ wavenumbers connected by that matrix element can differ from $k \cdot M = q$ by $Q_1$ in their first component.
Recall:

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Consider a matrix element for which \( Q_1 \neq 0 \). This means that the initial \((q)\) and final \((k)\) wavenumbers connected by that matrix element can differ from \( k \cdot \mathbb{M} = q \) by \( Q_1 \) in their first component.

Keeping only \( J_1 \), is it possible for a wavenumber to be mapped back onto itself by such a coupling? Seek solutions to

\[
(q_1 \quad q_2) \cdot \mathbb{M} = (q_1 + 1 \quad q_2) \quad \Longrightarrow \quad (q_1 \quad q_2) = (0 \quad 1).
\]

The matrix element connecting the \((0 \quad 1)\) mode to itself is

\[ T_{(0 \quad 1), (0 \quad 1)} = e^{-\epsilon} i J_1 (K). \]
Eigenfunction for $K = 0.3$ and $\epsilon = 10^{-3}$

(Renormalised by decay rate)

$i = 25$

$i = 30$
Decay Rate

For small $K$, the dominant Bessel function is $J_1$, so the decay factor $\mu^2$ for the variance is given by

$$\mu = |\mathbb{T}_{(0,1),(0,1)}| = e^{-\epsilon} J_1 (K) = \frac{1}{2} K + \mathcal{O}(\epsilon K, K^2).$$

Hence, for small $K$ the decay rate is limited by the $(0, 1)$ mode. The decay rate is independent of $\epsilon$ for $\epsilon \to 0$. 
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This is an analogous result to the baker’s map [Fereday et al., Wonhas and Vassilicos, PRE (2002)], except that here the agreement with numerical results is good for $K$ quite close to unity.
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This is because in the baker’s map the discontinuity generates many slowly-decaying harmonics at each step.
Decay Rate as $\epsilon \to 0$
Variance Spectrum of the Eigenfunction

- The long-wavelength mode $(0 \ 1)$ is the bottleneck that determines the decay rate, for small $K$. 
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- The long-wavelength mode \((0 \ 1)\) is the bottleneck that determines the decay rate, for small \(K\).
- But this dominant mode does not determine the structure of the eigenfunction.
- In fact, a very small amount of the total variance actually resides in that bottleneck mode: the variance is concentrated at small scales.
The wavenumbers are mapped back to themselves, with their variance decreased by a uniform factor $\mu^2 < 1$ (vertical arrows). But at the same time the modes are mapped to next one down the cascade following the diagonal arrows.
The decrease in variance for each of the diagonal arrows is diffusive and is given by the factor \( \nu_n = \exp(-2\epsilon k_n^2) \).
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If we denote by \( \sigma_n^{(i)} := |\hat{\theta}_{k_n}|^2 \) the variance in mode \( k_n \) at the \( i \)th iteration, we have

\[
\sigma_n^{(i)} = \mu^2 \sigma_n^{(i-1)}, \quad n = 0, 1, \ldots, \\
\sigma_n^{(i)} = \nu_{n-1} \sigma_{n-1}^{(i-1)}, \quad n = 1, 2, \ldots.
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\end{align*}
\]

These two recurrences can be combined to give

\[
\Sigma_n^{(i)} := \frac{\sigma_n^{(i)}}{\sigma_0^{(i)}} = \frac{\nu_{n-1} \nu_{n-2} \cdots \nu_0}{\mu^{2n}} = \mu^{-2n} \exp\left(-2\epsilon \sum_{m=0}^{n-1} k_m^2\right),
\]

where \( \Sigma_n^{(i)} \) is the relative variance in the \( n \)th mode.
The wavenumber is given by the exponential recursion,

$$\| k_n \| \simeq \Lambda \| k_{n-1} \| \quad \Rightarrow \quad \| k_n \| \simeq \Lambda^n \| k_0 \| = \Lambda^n .$$
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$$\|k_n\| \simeq \Lambda \|k_{n-1}\| \implies \|k_n\| \simeq \Lambda^n \|k_0\| = \Lambda^n.$$ 

Solve for $n = \log \|k_n\| / \log \Lambda$ and rewrite the relative variance as

$$\Sigma^{(i)}_n \simeq \|k_n\|^{-2\log \mu / \log \Lambda} \exp \left( -2\epsilon k^2_n / \Lambda^2 \right),$$

where we retained only the $k^2_{n-1}$ (last) term of the sum.
The wavenumber is given by the exponential recursion,

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Does not (and should not) depend on the iteration number, \( i \), and depends only on \( n \) through \( \mathbf{k}_n \). Find

\[ \Sigma(k) = k^{2\zeta} \exp \left( -2 \epsilon k^2 / \Lambda^2 \right) , \quad \zeta := -\log \mu / \log \Lambda, \]

the spectrum of relative variance.
Spectrum of Variance

\[ K = 10^{-3} \]
\[ \epsilon = 10^{-4} \]
\[ i = 12 \]
Conclusions

- Three phases of chaotic mixing: constant variance, superexponential decay, exponential decay.
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- Large $K$? Periodic orbits?