Going with the flow:
A study of Lagrangian derivatives

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12 February 2001

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Overview

- In a fluid flow, Lagrangian coordinates label fluid elements. The Lagrangian frame moves and stretches with the flow.
- Many equations of fluid dynamics simplify considerably in Lagrangian coordinates: the advection–diffusion equation, the magnetic induction equation...
- When the flow is chaotic [trajectories exhibit exponential separation], many quantities that characterize the geometry of the system have a well-defined asymptotic behavior when expressed in Lagrangian coordinates: Lyapunov exponents, characteristic directions...
If we represent a system in Lagrangian coordinates, we need to evaluate Lagrangian derivatives of physical quantities.

For example, in following the evolution of a magnetic field $B$ chaotically advected by a flow [the kinematic dynamo problem], to evaluate the induced current,

$$\mu_0 j = \nabla \times B$$

we need to know the Lagrangian derivatives of the frame in which $B$ is expressed. This requires the derivatives of the coefficients of expansion (the amount of stretching) and the characteristic directions (the axes of stretching).

The study of these Lagrangian derivatives leads to some surprising results: they must obey constraints due to the chaotic nature of the flow.
Lagrangian Coordinates

The trajectory of a fluid element in Eulerian coordinates $x$ satisfies

$$\frac{dx}{dt}(a, t) = v(x(a, t), t)$$

where $a$ are Lagrangian coordinates that label fluid elements. The usual choice is to take as initial condition $x(a, t = 0) = a$, which says that fluid elements are labeled by their initial position.

$x = x(a, t)$ is thus the transformation from Lagrangian ($a$) to Eulerian ($x$) coordinates.

For a chaotic flow, this transformation gets horrendously complicated as time evolves.
The Metric Tensor

The *Jacobian matrix* of the transformation \( \mathbf{x}(\mathbf{a}, t) \) is

\[
M^i_q \equiv \frac{\partial x^i}{\partial a^q}
\]

For simplicity, we restrict ourselves to incompressible flows, \( \nabla \cdot \mathbf{v} = 0 \), so that \( \det M = 1 \). The Jacobian matrix is a precise record of how a fluid element is *rotated* and *stretched* by \( \mathbf{v} \). We are interested in the stretching, not the rotation, so we construct the *metric tensor*

\[
g_{pq} \equiv \sum_{i=1}^{n} M^i_p M^i_q
\]

which contains only the information on the stretching of fluid elements.
Stretching and Contracting Directions

The metric is a symmetric, positive-definite matrix, so it can be diagonalized with orthogonal eigenvectors \( \{ \hat{e}_\sigma \} \) and corresponding real, positive eigenvalues \( \{ \Lambda^2_\sigma \} \),

\[
g_{pq} = \sum_{\sigma=1}^{n} \Lambda^2_\sigma \, (\hat{e}_\sigma)_p \, (\hat{e}_\sigma)_q
\]

The \( \Lambda_\sigma \) are called coefficients of expansion and are ordered such that \( \Lambda_1 > \Lambda_2 > \ldots > \Lambda_n \) [assumed nondegenerate].

The \( \Lambda_\sigma \) are related to the finite-time Lyapunov exponents \( \lambda_\sigma \) by

\[\lambda_\sigma = \log \Lambda_\sigma / t.\]

The incompressibility of \( v \) implies that \( \prod_\sigma \Lambda_\sigma = 1. \)
We use the label $u$ to indicate the most unstable direction:

$$\hat{e}_1 \equiv \hat{u}, \quad \Lambda_1 \equiv \Lambda_u$$

After some time, $\Lambda_u \gg 1$, growing exponentially for long times.

The label $s$ indicates the most stable direction:

$$\hat{e}_n \equiv \hat{s}, \quad \Lambda_n \equiv \Lambda_s$$

After some time, $\Lambda_s \ll 1$, decreasing exponentially for long times.

For autonomous flows $[v = v(x)$, not a function of $t$], there is always an intermediate direction, denoted by $m$, that does not grow or decrease exponentially [but may do so algebraically]. It corresponds to displacements along the trajectory of the system.
The eigenvalues and eigenvectors describe the deformation of a fluid element in a comoving frame:

The \( \mathbf{u} \) and \( \mathbf{s} \) directions can be integrated to yield the unstable and stable manifolds.
Here’s a portion of the stable manifold $s(a)$ for the $ABC$ 522 flow:
Singular Value Decomposition

It is difficult to study the Jacobian matrix $M = \partial \mathbf{x}/\partial \mathbf{a}$ directly because it mixes vastly different scales associated with the stretching and contracting directions.

Preferable to decompose $M$ as

$$M^i_q = \frac{\partial x^i}{\partial a^q} = \sum_\sigma (\hat{\mathbf{w}}_\sigma)^i \Lambda_\sigma (\hat{\mathbf{e}}_\sigma)_q$$

where $\hat{\mathbf{w}}_\sigma$ and $\hat{\mathbf{e}}_\sigma$ are orthonormal basis vectors. This decomposition is equivalent the singular value decomposition (SVD), with the $\Lambda_\sigma$ being the singular values.

The orthonormal vectors $\{\hat{\mathbf{e}}_\sigma\}$ gives the axes of stretching (strain) in Lagrangian space, and $\{\hat{\mathbf{w}}_\sigma\}$ gives the absolute orientation of a fluid element in Eulerian space.
The metric tensor in Lagrangian coordinates $g_{pq}$ can be written

$$g_{pq} = \sum_i M^i_p M^i_q = \sum_{i,\sigma,\tau} (\hat{e}_\sigma)_p \Lambda_\sigma (\hat{w}_\sigma)^i (\hat{w}_\tau)^i \Lambda_\tau \hat{e}_\tau)_q$$

$$= \sum_\sigma \Lambda_\sigma^2 (\hat{e}_\sigma)_p (\hat{e}_\sigma)_q ,$$

where we used the orthonormality of $\hat{w}_\sigma$.

This shows that $\Lambda_\sigma$ and $\hat{e}_\sigma$ are indeed the eigenvalues and eigenvectors of $g_{pq}$.

The SVD separates cleanly the parts of $M$ that are growing or shrinking exponentially in size (as determined by the coefficients of expansion $\Lambda_\sigma$).

Avoids the problems associated with evolving $M = \partial x / \partial a$ directly.
Greene and Kim (1987) derived the equations of motion for \( \hat{\mathbf{w}}_\sigma, \hat{\mathbf{e}}_\sigma, \) and \( \Lambda_\sigma \):

\[
\frac{d}{dt} \Lambda_\sigma = G_{\sigma \sigma} \Lambda_\sigma, \\
\hat{\mathbf{w}}_\tau \cdot \frac{d}{dt} \hat{\mathbf{w}}_\sigma = - \frac{G_{\tau \sigma} \Lambda^2_\sigma + G_{\sigma \tau} \Lambda^2_\tau}{\Lambda^2_\tau - \Lambda^2_\sigma} \quad \tau \neq \sigma; \\
\hat{\mathbf{e}}_\tau \cdot \frac{d}{dt} \hat{\mathbf{e}}_\sigma = - \frac{\Lambda_\tau \Lambda_\sigma}{\Lambda^2_\tau - \Lambda^2_\sigma} A_{\tau \sigma} \quad \tau \neq \sigma;
\]

where

\[
G_{\tau \sigma} \equiv \sum_{i,j} (\hat{\mathbf{w}}_\tau)^i \frac{\partial v^i}{\partial x^j} (\hat{\mathbf{w}}_\sigma)^j \quad A \equiv G + G^T.
\]

For nondegenerate coefficients of expansion \( \Lambda_\sigma \), these equations can be used to show that, in chaotic flows, the characteristic directions \( \hat{\mathbf{e}}_\sigma \) converge exponentially fast to constant values.
Lagrangian Derivative of the SVD

We can take the Lagrangian derivatives of the evolution equations for the components of the SVD. We obtain the asymptotic forms

\[ \Phi_{\kappa\mu\nu} = [(\hat{e}_\kappa \cdot \nabla_0)\hat{w}_\nu] \cdot \hat{w}_\mu = -\Phi_{\kappa\nu\mu} \sim \max (\Lambda_\kappa, \gamma_{\mu\nu}) \]

\[ \Psi_{\kappa\nu} = (\hat{e}_\kappa \cdot \nabla_0) \log \Lambda_\nu \sim \max (\Lambda_\kappa, 1) \]

\[ \Theta_{\kappa\mu\nu} = [(\hat{e}_\kappa \cdot \nabla_0)\hat{e}_\nu] \cdot \hat{e}_\mu = -\Theta_{\kappa\nu\mu} \sim \max (\gamma_{\mu\nu} \Lambda_\kappa, 1) \]

where \( \nabla_0 \equiv \partial / \partial a \), and

\[ \gamma_{\mu\nu} = \min \left( \frac{\Lambda_\mu}{\Lambda_\nu}, \frac{\Lambda_\nu}{\Lambda_\mu} \right) \ll 1, \quad \mu \neq \nu \]

\( \Phi \) and \( \Psi \) diverge along unstable directions at a rate \( \Lambda_\kappa \) [sensitive to initial conditions], and converge along stable directions.

\( \Theta \) has a more complicated behavior, but always diverges more slowly than \( \Phi \) and \( \Psi \). [Thiffeault, submitted to Nonlinearity.]
The Hessian

The **Hessian** is the quadratic form of second derivatives of \( x(a, t) \). Since \( M = \partial x/\partial a \), we can write

\[
\frac{\partial^2 x^i}{\partial a^p \partial a^q} = \frac{\partial M^i_p}{\partial a^q} = \frac{\partial M^i_q}{\partial a^p}.
\]

The asymptotic evolution of the Hessian is characterized by **generalized** or **higher-order** Lyapunov exponents, and was first studied by Dressler and Farmer (1992) and Taylor (1993).

We define a “projected” version of the Hessian,

\[
K^\kappa_{\mu\nu} \equiv \sum_{i,p,q} (\hat{\omega}_\kappa)^i \frac{\partial^2 x^i}{\partial a^p \partial a^q} (\hat{e}_\mu)_p (\hat{e}_\nu)_q
\]

with \( K^\kappa_{\mu\nu} = K^\kappa_{\nu\mu} \).
By differentiating the SVD directly, we have

\[ K_{\mu\nu}^\kappa = \Lambda_\kappa \Psi_{\mu\kappa} \delta_{\nu\kappa} + \Lambda_\kappa \Theta_{\mu\nu\kappa} + \Lambda_\nu \Phi_{\mu\kappa\nu}. \]

But since \( K_{\mu\nu}^\kappa \) is symmetric in \( \mu \) and \( \nu \), we could have equally well written

\[ K_{\mu\nu}^\kappa = \Lambda_\kappa \Psi_{\nu\kappa} \delta_{\mu\kappa} + \Lambda_\kappa \Theta_{\nu\mu\kappa} + \Lambda_\mu \Phi_{\nu\kappa\mu}, \]

Equating the two forms, we find

\[ \Lambda_\mu (\Theta_{\mu\mu\nu} + \Psi_{\nu\mu}) = \Lambda_\nu \Phi_{\mu\mu\nu}, \quad \mu \neq \nu, \]

\[ \Lambda_\kappa (\Theta_{\mu\nu\kappa} - \Theta_{\nu\mu\kappa}) = \Lambda_\nu \Phi_{\mu\nu\kappa} - \Lambda_\mu \Phi_{\nu\mu\kappa}, \quad \mu, \nu, \kappa \text{ differ.} \]

These relations allow us to solve for the \( \Phi \) in terms of the \( \Theta \) and \( \Psi \). However, in a chaotic system this inversion is highly singular, and so is not very useful.
The relations can be used in other ways. Since we are not interested in the $\Phi$’s [Eulerian], we substitute their asymptotic form and rewrite the relations as

$$
\Theta_{\mu\nu\mu} + \Psi_{\nu\mu} \sim \max \left( \Lambda_\nu, \gamma_{\mu\nu}^2 \right), \quad \mu < \nu, \quad (1)
$$

$$
\Theta_{\mu\nu\kappa} - \Theta_{\nu\mu\kappa} \sim \max \left( \frac{\Lambda_\mu \Lambda_\nu}{\Lambda_\kappa}, \frac{\Lambda_\mu^2}{\Lambda_\kappa^2} \right), \quad \kappa < \mu < \nu. \quad (2)
$$

When $\Lambda_\nu$ corresponds to a contracting direction, (1) goes to zero asymptotically.

When $(\Lambda_\mu \Lambda_\nu)/\Lambda_\kappa \to 0$, (2) goes to zero asymptotically.

These are constraints on the asymptotic form of the $\Psi$’s and $\Theta$’s.
The first type of constraint implies that in any chaotic flow

\[ \nabla_0 \cdot \hat{s} - \hat{s} \cdot \nabla_0 \log \Lambda_s \rightarrow 0, \]

where \( s \) denotes a contracting direction and \( \nabla_0 \) denotes a gradient with respect to Lagrangian coordinates.

This constraint was first derived for 2D flows by Tang and Boozer (1996) by requiring that the Riemann curvature tensor vanish for incompressible chaotic flows.

The constraint is valid in any number of dimensions, and can be derived regardless of the curvature. The constraint has important consequences for the advection–diffusion equation [Thiffeault, in preparation]
The other type of constraint we obtain is that for any chaotic flow (specializing to 3D)

\[ \Theta_{231} - \Theta_{321} = \hat{u} \cdot \nabla_0 \times \hat{u} \longrightarrow 0. \]

This implies that locally we can write

\[ \hat{u} = \frac{\nabla_0 \varphi}{\| \nabla_0 \varphi \|} \]

for some scalar \( \varphi \).

This constraint was also derived using curvature arguments by Thiffeault and Boozer (Chaos, 2001).

The global consequences of this constraint have yet to be investigated.
\[ t^u \]

\[ r_0 \]

\[ \hat{u} \cdot \nabla_0 \times \hat{u} \]

\[ (\Lambda_m/\Lambda_u)^2 \]

\[ t \]
Magnetic Field Evolution

The evolution of a magnetic field in resistive MHD is governed by the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}$$

where the Eulerian velocity field $\mathbf{v}(\mathbf{x}, t)$ is some prescribed time-dependent flow. $\mathbf{B}$ is the magnetic field, $\eta$ is the resistivity, and $\mu_0$ is the permeability of free space.

In a chaotic flow, fluid elements are stretched exponentially. The magnetic field grows due to the stretching, and the diffusion is also increased by this process. This enhancement is known as chaotic mixing.
Induction Equation in Lagrangian Coordinates

With the help of the metric tensor, we can transform the magnetic induction equation to Lagrangian coordinates $a$:

$$\frac{\partial}{\partial t} b^r \bigg|_a = \sum_{p,q=1}^{3} \frac{\eta}{\mu_0} \frac{\partial}{\partial a^p} \left[ g^{pq} \frac{\partial}{\partial a^q} b^r \right]$$

where $b^r(a, t) \equiv \sum_i (M^{-1})^r_i B^i$ is the magnetic field in the Lagrangian frame, and $g^{pq}(a, t) \equiv (g^{-1})^{pq}$. The above equation is a diffusion equation with an anisotropic and inhomogeneous diffusivity, $\eta g^{pq}$. By construction, the velocity $v$ has dropped out of the equation entirely.

When $\eta = 0$, the above is the well-known result that in ideal MHD the magnetic field is frozen into the fluid.
Constraints for the Dynamo

Let $\mu_0 j^u$ be the projection of $\mu_0 j = \nabla_0 \times B$ in Lagrangian coordinates along the unstable direction $\hat{u}$:

$$
\mu_0 j^u = \Lambda_u^2 b^u \cdot \nabla_0 \times \hat{u} + O(\Lambda_m^2).
$$

If not for the constraint $\hat{u} \cdot \nabla_0 \times \hat{u} \sim (\Lambda_m/\Lambda_u)^2 \to 0$, $j^u$ would go as $\Lambda_u^2$. After taking the constraint into account, we find

$$
\frac{j^2}{B^2} = \frac{(j \cdot B)^2}{B^2} \sim \Lambda_u^2, \quad j^2 \sim \Lambda_u^4
$$

so that $j \parallel \gg j \perp$. The opposite would be true without the constraint. This is the case in any “generic” chaotic flow.

Hence, the magnetic field is perpendicular to the current in the case of ideal evolution.
Summary

- **Lagrangian coordinates** are a powerful tool for studying chaotic flows.

- To evaluate quantities involving derivatives (e.g., $\nabla \times \mathbf{B}$), the **Lagrangian derivatives** of the coefficients of expansion $\Lambda_\sigma$ and of the **characteristic directions** $\hat{\mathbf{e}}_\sigma$ are needed.

- The asymptotic behavior of the Lagrangian derivatives can be obtained by **differentiating the SVD method** directly.

- These derivatives are not all independent and must obey **constraints** due to the exponential behavior in chaotic flows.

- The constraints have consequences in physical problems. For example, $\hat{\mathbf{u}} \cdot \nabla_0 \times \hat{\mathbf{u}} \to 0$ tells us that the **induced current** $\mathbf{j}$ is perpendicular to $\mathbf{B}$ for the kinematic dynamo.