Hyperbolic geometry  (Jean-Luc Thiffeault)

A plane is flat:

\[ \alpha + \beta + \gamma = \pi \]

A triangle on a plane has angles that sum to \( \pi \) (or 180°).

This is a property of Euclidean geometry.

Now let’s consider the surface of a sphere:

First, how do we draw a "straight line"? We define it as the shortest distance between points.
On a sphere, the shortest distance is given by the unique great circle through two points, except for antipodal points.

The technical name for such a straight line is a geodesic.

This is why flight routes seem funny when looking at a map: we tend to think of east-west as the "shortest distance", but it's not!
We can draw a triangle made up of geodesics:

For a sphere of radius 1, the triangle will have area:

\[ A = (\alpha + \beta + \gamma) - \pi \geq 0 \]

The sum of the interior angles is \( \geq \pi \).

This is one feature of non-Euclidean geometry.

What about a cylinder?

Looks "curved", but if we cut, we get a flat piece of paper, so angle sum to \( \pi \)!

Somehow the cylinder is "not curved"
How to measure curvature?

You might have seen a tangent line to a curve:

A tangent line touches a line at one point: the slope of the line = derivative at that point.

We can also draw a circle which is tangent, but such that its second derivative also agrees with the curve at point P.

This is called the osculating circle at point P.

The circle has a radius $R$.

$\kappa = \frac{1}{R}$ is called the curvature of the curve at P.
Now for a surface: "cut" along a plane:

We compute the curvature $k$ at $P$.

As we rotate the plane at $P$, we can show that there will be a maximum and minimum of $k$: $k_1 = \max k$, $k_2 = \min k$

Define: $K = k_1 k_2$ \textbf{Gaussian curvature}

One detail: we assign a different sign depending on which side of the surface the osculating circle lies.

$K \neq 0$ means the surface is \textbf{intrinsically curved}
Sphere of radius $R$ has Gaussian curvature

$$K = \frac{1}{R^2}$$

at every point.

Cylinder has

$$k_1 = \frac{1}{R}, \quad k_2 = 0 \implies \text{since } K = 0 \text{ for a straight line}$$

So $K = 0$ everywhere!

Are there surfaces with $K < 0$?

Saddle (Pringles potato chip).

$$K < 0$$

But the curvature is not the same everywhere.
There is a special curve called a **tractrix**

\[ y(x) = \text{sech}^{-1} x - \sqrt{1 - x^2} \]

\( \text{sech} = \) "hyperbolic secant"

(similar to secant, ... but different. Not important for us)
If we spin the tractrix along the $y$ axis, we get a surface of revolution called a pseudosphere or tractricoid.

This surface has constant negative curvature $K = -1$.

A surface with negative curvature everywhere is called hyperbolic.

If we draw a triangle with geodesic sides, the angles are less than $\pi$:

$$\alpha + \beta + \gamma < \pi$$

which is a feature of a surface of negative curvature.
Why is it called the pseudosphere?

This has the same area and volume as a sphere of radius 1!

In fact since it is hard to draw triangles on a curved surface, we use the Poincaré disk model:

Straight lines are defined as semicircles perpendicular to boundary!
This model turns out to have constant Gaussian curvature $= -1$.

The Poincaré disk has many fascinating properties, such that it can be tiled with any regular polygon (unlike the flat plane).

See famous pictures by M. C. Escher!