Lyapunov Exponents and Transport in 2D Flows

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18 November 1999

with Allen Boozer
Overview

We are interested in the advection-diffusion equation:

\[
\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi = \frac{1}{\rho} \nabla \cdot (\rho D \nabla \phi)
\]

where the Eulerian velocity field \( \mathbf{v}(\mathbf{x}, t) \) is some prescribed time-dependent flow, which may or may not be be chaotic. The quantity \( \phi \) represents the concentration of some passive scalar, \( \rho \) is the density, and \( D \) is the diffusion coefficient.

We assume that the Lagrangian dynamics are strongly chaotic \((\lambda L^2/D \gg 1)\).
Lagrangian Coordinates

The trajectory of a fluid element in Eulerian coordinates $x$ satisfies

$$\frac{dx}{dt}(\xi, t) = v(x(\xi, t), t),$$

where $\xi$ are Lagrangian coordinates which label fluid elements. The usual choice is to take as initial condition $x(\xi, t = 0) = \xi$, which says that fluid elements are labeled by their initial position.

$x = x(\xi, t)$ is thus the transformation from Lagrangian ($\xi$) to Eulerian ($x$) coordinates.

This transformation gets horrendously complicated as time evolves.
Lyapunov Exponents

The rate of exponential separation of neighbouring Lagrangian trajectories is measured by Lyapunov exponents

$$\lambda_\infty = \lim_{t \to \infty} \frac{1}{t} \ln \| (T_x v) w_0 \|,$$

where $T_x v$ is the tangent map of the velocity field (the matrix $\frac{\partial v}{\partial x}$) and $w_0$ is some constant vector.

Lyapunov exponents converge very slowly. So, for practical purposes we are always dealing with finite-time Lyapunov exponents.
The Idea

- Can we characterize the spatial and temporal evolution of finite-time Lyapunov exponents in a generic manner?
- Can we quantify the impact of these exponents on diffusion?

Tang and Boozer (1996) brought the tools of differential geometry to bear on this problem.

**Results:** a generic functional form for the time evolution of finite-time Lyapunov exponents, and a relation between their spatial dependence and the shape of the stable manifolds.
A little differential geometry ...

The Jacobian of the transformation from Lagrangian ($\xi$) to Eulerian ($\mathbf{x}$) coordinates

$$J^i_j \equiv \frac{\partial x^i}{\partial \xi^j}$$

The Jacobian tells us how tensors transform:

- **Covariant:**
  $$\tilde{V}_j = J^k_j \, V_k,$$

- **Contravariant:**
  $$\tilde{W}^i = J^i_k \, W^k.$$
Measuring distances

The distance between two infinitesimally separated points in Eulerian space is given by

\[ ds^2 = d\mathbf{x} \cdot d\mathbf{x} = \delta_{ij} \, dx^i \, dx^j . \]

Therefore, in Lagrangian coordinates distances are given by

\[ ds^2 = \delta_{ij} \left( \frac{dx^i}{d\xi^k} \right) \left( \frac{dx^j}{d\xi^\ell} \right) = (J^i_k \, \delta_{ij} \, J^j_\ell) \, d\xi^k \, d\xi^\ell . \]

The distance function now depends on the Lagrangian coordinate \( \xi \) through the Jacobian \( J \).
The Metric Tensor

The tensor $\delta_{ij}$ is a metric in the Eulerian (Euclidean) space. The tensor

$$g_{k\ell}(\mathbf{x}, t) \equiv \sum_i J^i_k \ J^i_\ell = (J^T J)_{k\ell}$$

is the same metric tensor but in the Lagrangian coordinate system. Since the metric tells us about the distance between two neighbouring Lagrangian trajectories, its eigenvalues are related to the finite-time Lyapunov exponents.
2-D Incompressible Flow

We will now restrict ourselves to a 2-D, incompressible velocity field $\mathbf{v}$. This means that

$$\det g = (\det J)^2 = 1.$$ 

Now, $g$ is a positive-definite symmetric matrix, which implies that it has real positive eigenvalues, $\Lambda(\xi, t) \geq 1$ and $\Lambda^{-1}(\xi, t) \leq 1$, and orthonormal eigenvectors $\hat{e}(\xi, t)$ and $\hat{s}(\xi, t)$:

$$g_{k\ell}(\xi, t) = \Lambda e_k e_\ell + \Lambda^{-1} s_k s_\ell$$

The finite-time Lyapunov exponents are given by

$$\lambda(\xi, t) = \ln \Lambda(\xi, t) / 2t$$
Stable and Unstable Directions

At a fixed coordinate $\xi$:

The stable and unstable manifolds $\hat{e}(\xi, t)$ and $\hat{s}(\xi, t)$ converge exponentially to their asymptotic values $\hat{e}_\infty(\xi)$ and $\hat{s}_\infty(\xi)$, whereas Lyapunov exponents converge logarithmically.
Model System

Oscillating convection rolls: $\mathbf{v} = (-\partial_y \psi, \partial_x \psi)$, with

$$
\psi(x, t) = Ak^{-1}(\sin kx \sin \pi y + \epsilon \cos \omega t \cos kx \cos \pi y)
$$
$\hat{s}_\infty$ field for oscillating rolls with $A = k = \epsilon = \omega = 1$, with two typical portions of the stable manifold in red and blue.
The Advection-diffusion Equation

Under the coordinate change to Lagrangian variables the diffusion term becomes

\[ \nabla \cdot (D \nabla \phi) = \frac{\partial}{\partial x^i} (D \delta^{ij} \frac{\partial \phi}{\partial x^j}) = \frac{\partial}{\partial \xi^i} (D g^{ij} \frac{\partial \phi}{\partial \xi^j}). \]

In Lagrangian coordinates the diffusivity becomes \( D g^{ij} \): it is no longer isotropic.

The advection-diffusion equation is thus just the diffusion equation,

\[ \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial \xi^i} (D g^{ij} \frac{\partial \phi}{\partial \xi^j}), \]

because by construction the advection term drops out.
Diffusion along $\hat{s}_\infty$ and $\hat{e}_\infty$

The diffusion coefficients along the $\hat{s}_\infty$ and $\hat{e}_\infty$ lines are

$$D^{ss} = s_\infty^i (Dg^{ij}) s_\infty^j = D \exp(2\lambda t),$$

$$D^{ee} = e_\infty^i (Dg^{ij}) e_\infty^j = D \exp(-2\lambda t).$$

We see that $D^{ee}$ goes to zero exponentially quickly, while $D^{ss}$ grows exponentially.

Hence, essentially all the diffusion occurs along the $\hat{s}_\infty$-line.
**Spatial Dependence of $\lambda(\xi, t)$**

Differential geometry tells us if a metric describes a flat space, then its Riemann curvature tensor must vanish in every coordinate system.

After some tedious algebra, we find this implies that the quantity

$$\hat{s}_\infty \cdot \nabla_0 \lambda(\xi, t) t + \nabla_0 \cdot \hat{s}_\infty$$

converges to 0 exponentially. Hence, it can be shown that the finite-time Lyapunov exponents must have the form

$$\lambda(\xi, t) = \frac{\tilde{\lambda}(\xi)}{t} + \frac{f(\xi, t)}{\sqrt{t}} + \lambda_\infty,$$

where $\hat{s}_\infty \cdot \nabla_0 f = 0$ (the $1/\sqrt{t}$ factor comes from known results on the variance of the exponents).
Example:

\[ \langle \lambda \rangle = 0.305/t + 0.175/\sqrt{t} + 0.117 \]

Dotted: Numerical
Solid: \[ 0.305/t + 0.175/\sqrt{t} + 0.117 \]

Allows us to determine \( \lambda_\infty = 0.117 \) rapidly and accurately.
Convergence on the $\hat{s}_\infty$-line

$$\nabla_0 \cdot \hat{s}_\infty + (\hat{s}_\infty \cdot \nabla_0) \lambda t$$ evaluated on an $\hat{s}_\infty$-line.

$\tau$ is the distance along the red $\hat{s}_\infty$-line on page 12.

Green: $-\nabla_0 \cdot \hat{s}_\infty$

Red: $(\hat{s}_\infty \cdot \nabla_0) \lambda t$. 
Curvature and Lyapunov Exponents

Finite-time Lyapunov exponent $\lambda(\xi(\tau), t)$ has local minima near high-curvature $\kappa \equiv (\mathbf{s}_\infty \cdot \nabla_0)\mathbf{s}_\infty$ regions of $\mathbf{s}_\infty$-line.
Conclusions

- Diffusion occurs overwhelmingly along the stable direction.
- The spatial dependence of Lyapunov exponents along $\hat{s}$ lines is contained in the smooth function $\tilde{\lambda}(\xi)$, which decays as $1/t$.
- The notoriously slow convergence of Lyapunov exponents is embodied in the nonsmooth function $f(\xi, t)$, which is \textit{constant} on $\hat{s}$ lines and decays as $1/\sqrt{t}$.
- Relationship between $\hat{s}_\infty(\xi), \kappa \equiv (\hat{s}_\infty \cdot \nabla_0)\hat{s}_\infty$, and $\tilde{\lambda}(\xi)$.
- Sharp bends in the $\hat{s}$ line lead to locally small finite-time Lyapunov exponents (diffusion is hindered).
- Tested directly on oscillating-rolls flow.