Long-wave Instability in Anisotropic Double-Diffusion

Jean-Luc Thiffeault

Institute for Fusion Studies
and Department of Physics
University of Texas at Austin

and

Neil J. Balmforth

Department of Theoretical Mechanics
University of Nottingham

November 1997

APS DFD-97
Overview

- Want to capture asymptotic dynamics near Takens–Bogdanov bifurcation.

- Problem: typical scaling leads to Hamiltonian (and thus conservative) equation, which obviously does not capture a lot of the dynamics.

- Try using different scaling, but then get unremovable resonant terms.

- **Solution:** extend parameter space to allow removal of resonant terms. Raises codimension, but asymptotic.
Takens–Bogdanov

- TB bifurcation occurs when two modes become unstable at the same parameter values.

- Equations for the reduced dynamics near this bifurcation point capture more of the diverse behaviour of the system than simple steady or Hopf bifurcation.

- For double-diffusive convection in long-wave theory such a bifurcation is present.

- Problem: the reduced equations contain terms of differing order in the standard asymptotic expansion parameter. The asymptotic theory fails to collect a dissipative nonlinear term; the amplitude equations are Hamiltonian to leading order (Childress and Spiegel 1981).
Possible solutions

- **Normal form theory**: not available for extended (continuum of excited modes) systems.

- **Reconstitution**: Not asymptotic, so hard to judge validity. May be flawed in some cases (Clune, Depassier, and Knobloch, 1994).

- **Nonlocal averaging**: Difficult to solve (Pismen, 1988).

- Alternative route: if more parameters were available, could remove resonant terms at the cost of augmenting the codimension of the bifurcation.

To introduce needed extra parameters, we choose **anisotropic** double-diffusion as our system. (possible transport model for ocean, astrophysics, tokamak plasmas)
Illustration of Procedure

Normal form for three real marginal modes:

\[
\begin{align*}
\dot{F} &= G \\
\dot{G} &= H \\
\dot{H} &= -\eta H - \nu G - \lambda F + a F^2 + b G^2 + c F G + d F H
\end{align*}
\]

Assuming strongly damped mode \((|\eta| >> |\nu|, |\lambda|)\) we should recover the two-mode normal form. One way to do this (Spiegel et al) is to use the scaling

\[
t = \tilde{t}/\delta, \quad \lambda = \delta^2 \bar{\lambda}, \quad \nu = \delta^2 \bar{\nu}, \quad F = \delta^2 \bar{F}, \quad G = \delta^3 \bar{G}.
\]

This leads to a Hamiltonian equation, not two-mode normal form as one would expect. If instead of rescaling the amplitudes one rescales the nonlinear terms

\[
t = \tilde{t}/\delta, \quad \lambda = \delta^2 \bar{\lambda}, \quad \nu = \delta \bar{\nu}, \quad a = \delta^2 \bar{a}, \quad c = \delta \bar{c},
\]

we recover two-mode normal form, at the cost of raising the codimension.
Model Equations

The equations for anisotropic double-diffusion are

\[
\begin{align*}
\sigma^{-1} \frac{d}{dt} \nabla^2 \psi & = R \partial_x \Theta - S \partial_x \Sigma + (D^2 + \Lambda \partial_x^2) \nabla^2 \psi, \\
\frac{d}{dt} \Theta & = \partial_x \psi + (D^2 + \Lambda \partial_x^2) \Theta, \\
\text{Le} \frac{d}{dt} \Sigma & = \text{Le} \partial_x \psi + (D^2 + \Xi \partial_x^2) \Sigma;
\end{align*}
\]

with no-slip, fixed-flux boundary conditions

\[
\psi = D\psi = 0, \quad D \Theta = D \Sigma = 0, \quad z = 0 \text{ and } 1
\]

Fixed flux favors convection cells that are as large as the system will permit. Use this to define small parameter \( \epsilon \).

Scaling:

\[
\partial_x = \epsilon \partial_X, \quad \partial_t = \epsilon^4 \partial_T, \quad \psi = \epsilon \phi_X
\]
Order $\epsilon^0$ and $\epsilon^2$

The fixed flux boundary conditions give

$$\Theta_0 = \Theta_0(X, T), \quad \Sigma_0 = \Sigma_0(X, T)$$

at order $\epsilon^0$.

At order $\epsilon^2$, we get the solvability condition (linear at this order):

$$\begin{pmatrix}
\frac{1}{720} R_0 - \Lambda_0 & -\frac{1}{720} S_0 \\
\frac{1}{720} \text{Le}_0 R_0 & -\frac{1}{720} \text{Le}_0 S_0 - \Xi_0
\end{pmatrix}
\begin{pmatrix}
\Theta_{0XX} \\
\Sigma_{0XX}
\end{pmatrix} = 0.$$

The requirement that the matrix have zero eigenvalues means that its trace and determinant must vanish. This is obtained by letting

$$R_0 = 720 \frac{\Lambda_0^2}{\Lambda_0 - \Xi_0/\text{Le}_0}, \quad S_0 = 720 \frac{\Xi_0^2}{\text{Le}_0^2 \Lambda_0 - \Xi_0/\text{Le}_0},$$

The eigenvector for the matrix is parametrized by $\Sigma_0 = (\text{Le}_0 \Lambda_0/\Xi_0) \Theta_0$ (it only has one).
Order $\epsilon^4$

Get two solvability conditions again, this time involving $T$:

$$\Theta_{0T} = \ldots$$
$$\Sigma_{0T} = (Le_0 \Lambda_0 / \Xi_0) \Theta_{0T} = \ldots$$

Must be compatible since $\Theta_{0T}$ and $\Sigma_{0T}$ are related. This is not satisfied automatically; this is why we now make use of the extra parameters. By letting

$$Le_0 = 1$$
$$5(\Lambda_0 + \Xi_0) = 11(1 + \Lambda_0)$$
$$R_2 - \frac{\Lambda_0}{\Xi_0} S_2 = \frac{720\Lambda_0(\Lambda_2 - \Xi_2 + Le_2 \Xi_0)}{\Lambda_0 - \Xi_0}$$

the two become compatible. This increases the codimension by three.
Marginal Stability Curves

\[ \Delta_0 = 0.1 \text{ (long-dashed), } 1.2727 \text{ (solid), } 5 \text{ (dashed)} \]
We get a solvability condition involving only the $\epsilon^2$ integration constants

$$g(X, T) := \Sigma_{2,0}(X, T) - \frac{\Lambda_0}{\Xi_0} \Theta_{2,0}(X, T)$$

at this order. After rescaling to eliminate some parameters we have the coupled system

\begin{align*}
    f_T &= g_{XX} + \alpha f_{XX} + f_{XXXX} + (f^3_X)_X \\
    g_T &= \lambda f_{XX} + \kappa f_{XXXX} - \gamma f_{XXXXXX} + \beta g_{XX} \\
    &\quad - \rho g_{XXXX} + \xi (f^3_X)_X + (f^2_X g_X)_X \\
    &\quad + \eta (f_X f^2_{XX})_X - \zeta (f^3_X)_{XXX}
\end{align*}

We fixed $Le_0$, $\Delta_0$, and $\Lambda_2$. However, we are left with enough parameters to vary independently all the coefficients except $\eta$ and $\zeta$. 

APS DFD-97
Captures Turnaround

Steady state

Blue $- A|f|^3$
Red $- A|f|^3 + B|f|^5$
Numerical Solution

Traveling waves stable.
Conclusions

- For anisotropic double-diffusion in long-wave theory, we have shown that an extended system equation can be asymptotically derived.


- Compare reconstituted result.

- Explore numerical solutions.

- Make connection with physics.