optimizing transport in heat exchangers
a probabilistic approach

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Advection and diffusion of heat in a bounded region $\Omega$, with Dirichlet boundary conditions:

$$\partial_t \theta + u \cdot \nabla \theta = D \Delta \theta, \quad u \cdot \hat{n} |_{\partial \Omega} = 0, \quad \theta |_{\partial \Omega} = 0,$$

with $\nabla \cdot u = 0$ and $\theta(x, t) \geq 0$.

Write $\langle \cdot \rangle$ for an integral over $\Omega$. The rate of heat loss is equal to the flux through the boundary $\partial \Omega$:

$$\partial_t \langle \theta \rangle = D \int_{\partial \Omega} \nabla \theta \cdot \hat{n} \, dS =: -F[\theta] \leq 0.$$  \[\text{Goal: find velocity fields } u \text{ that maximize the heat flux.}\]

Note that * is not so good for this, since velocity does not appear.
related problem: mean exit time

Take steady velocity $u(x)$. The mean exit time $\tau(x)$ of a Brownian particle initially at $x$ satisfies

$$-u \cdot \nabla \tau = D \Delta \tau + 1, \quad \tau|_{\partial \Omega} = 0,$$

This is a steady advection–diffusion equation with velocity $-u$ and source 1.

Intuitively, a small integrated exit time $\langle \tau \rangle = \|\tau\|_1$ implies that the velocity is good at taking heat out of the system.

The exit time equation is much nicer than the equation for the concentration: it is steady, and it applies for any initial concentration $\theta_0(x)$. 

Recall that $\langle \cdot \rangle$ is an integral over space, and take $\langle \theta_0 \rangle = 1$. The quantity

$$
\int_0^\infty \langle \theta \rangle \, dt
$$

is a cooling time. Smaller is better for transport.

We have the rigorous bounds

$$
\int_0^\infty \langle \theta \rangle \, dt \leq \|\tau\|_\infty \quad \int_0^\infty \langle \theta \rangle \, dt \leq \|\tau\|_1 \|\theta_0\|_\infty.
$$

Thus, decreasing a norm like $\|\tau\|_1$ or $\|\tau\|_\infty$ will typically decrease the cooling time, as expected.
optimization problem

Advection–diffusion operator and its adjoint:

\[ \mathcal{L} := u \cdot \nabla - D \Delta, \quad \mathcal{L}^\dagger = -u \cdot \nabla - D \Delta. \]

Minimize \( \langle \tau \rangle \) over steady \( u(x) \) with fixed total kinetic energy \( E \).

The functional to optimize:

\[ \mathcal{F}[\tau, u, \vartheta, \mu, p] = \langle \tau \rangle - \langle \vartheta (\mathcal{L}^\dagger \tau - 1) \rangle + \frac{1}{2} \mu (\|u\|_2^2 - 2E) - \langle p \nabla \cdot u \rangle \]

Here \( \vartheta, \mu, p \) are Lagrange multipliers.
Simple system: 2D disk. Think of the cross-section of a pipe.

For small energy $E$, exact solution in terms of Bessel functions $J_m(\rho_n)$, where $\rho_n$ are zeros.

Pick the solution with largest transport: $m = 2, n = 1$: 
asymptotics: large E case

Numerical solution with **bvp5c** (Shampine, 2000), using a continuation method.

Asymptotics at large E, fixed m: $\langle \tau \rangle \sim m^{-2/3} E^{-1/3}$. 
asymptotics: large $E$ case (cont’d)

Optimal $m$ at fixed energy $E$: 

![Graph showing the optimal $m$ at fixed energy $E$. The graph plots $E$ on the x-axis and $\frac{\Lambda}{\nu}$ on the y-axis, with different lines representing various $m$ values. The graph highlights the case $m = 32$.](image)
Penalty on large $m$: the "stagnation zone"
structure of the solution for large $E$
Transport in heat exchangers has a very different character than ‘freely-decaying’ problem.

Using the probabilistic mean exit time formulation simplifies the problem. (Idea came from Iyer et al. 2010.)

Optimal solutions for $u$ are reminiscent of Dean flow.

Optimal exit time at fixed flow energy shows increasing number of “cells” as energy increased.

This is a pathology of fixing $E$. In future work we will fix viscous dissipation, which penalizes small structures.

Generalizations: use different norms, spatial weight...