A Topological Theory of Stirring

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Figure-eight stirring protocol

- Classic stirring method!
- Viscous (Stokes) flow;
- Essentially two-dimensional;
- Two regular islands: there are effectively 3 rods!
- We call these Ghost Rods
- ‘Injection’ from the top;
- Dye (material line) stretched exponentially.

Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 1]
Channel flow

Experiments by E. Gouillart and O. Dauchot (CEA Saclay).

[movie 2] [movie 3]
Channel flow: Injection

- Four-rod stirring device, with two ghost rods;
- Channel flow is upwards;
- Direction of rotation matters a lot!
- This is because it changes the injection point.
- Flow breaks symmetry.

Goals:
- Connect features to topology of rod motion: stretching rate, injection point, mixing region;
- Use topology to optimise stirring devices.
Mathematical description

Periodic stirring protocols in two dimensions can be described by a **homeomorphism** \( \varphi : S \to S \), where \( S \) is a compact orientable surface.

For instance, in the previous slides,
- \( \varphi \) describes the mapping of fluid elements after one full period of stirring, obtained from solving the Stokes equation;
- \( S \) is the **disc** with holes in it, corresponding to the stirring rods.

Task: **Categorise all possible** \( \varphi \).

\( \varphi \) and \( \psi \) are **isotopic** if \( \psi \) can be continuously ‘reached’ from \( \varphi \) without moving the rods. Write \( \varphi \simeq \psi \).
Thurston–Nielsen classification theorem

\( \varphi \) is isotopic to a homeomorphism \( \varphi' \), where \( \varphi' \) is in one of the following three categories:

1. **finite-order**: for some integer \( k > 0 \), \( \varphi'^k \simeq \text{identity} \);
2. **reducible**: \( \varphi' \) leaves invariant a disjoint union of essential simple closed curves, called *reducing curves*;
3. **pseudo-Anosov**: \( \varphi' \) leaves invariant a pair of transverse measured singular foliations, \( \mathcal{F}^u \) and \( \mathcal{F}^s \), such that \( \varphi'(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda \mu^u) \) and \( \varphi'(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1} \mu^s) \), for dilatation \( \lambda \in \mathbb{R}_+ \), \( \lambda > 1 \).

The three categories characterise the isotopy class of \( \varphi \).

Number 3 is the one we want for good mixing.
What’s a foliation?

• A pseudo-Anosov (pA) homeomorphism stretches and folds a bundle of lines (leaves) after each application.

• This bundle is called the unstable foliation, $\mathcal{F}_u$.

• Arcs are measured by ‘counting’ the number of leaves crossed.

• Two arcs transverse to a foliation $\mathcal{F}$, with the same transverse measure.

• If we iterate $\varphi$, the transverse measure of these arcs increases by a factor $\lambda$. 
A singular foliation

The ‘pseudo’ in pseudo-Anosov refers to the fact that the foliations can have a finite number of **pronged singularities**.

3-pronged singularity

Boundary singularity

But do these things exist?
Visualising a singular foliation

- A four-rod stirring protocol;
- Material lines trace out leaves of the unstable foliation;
- One 3-pronged singularity.
- One injection point (top): corresponds to boundary singularity;
Thurston introduced **train tracks** as a way of characterising the measured foliation. The name stems from the ‘cusps’ that look like train switches.
What are train tracks good for?

- They tell us the possible types of measured foliations.
- **Exterior cusps** correspond to boundary singularities.

![Diagram showing train tracks with interior and exterior cusps, 3-pronged singularities, and injection points.]

- These exterior cusps are the injection points.
- For three rods, only one type!
- The stirring protocol gives the **train track map**.
- Stokes flow reproduces these features remarkably well.
Train track map for figure-eight

1 → a → 2 → b → 3
Train track map: symbolic form

\[ a \mapsto a \bar{2} \bar{a} \bar{1} a b \bar{3} \bar{b} \bar{a} \bar{1} a, \quad b \mapsto \bar{2} \bar{a} \bar{1} a b \]

Easy to show that this map is efficient: under repeated iteration, cancellations of the type \( a \bar{a} \) or \( b \bar{b} \) never occur.

There are algorithms, such as Bestvina & Handel (1992), to find efficient train tracks. (Toby Hall has an implementation in C++)
As the TT map is iterated, the number of symbols grows exponentially, at a rate given by the topological entropy, \( \log \lambda \). This is a lower bound on the minimal length of a material line caught on the rods.

Find from the TT map by Abelianising: count the number of occurrences of \( a \) and \( b \), and write as matrix:

\[
\begin{pmatrix}
  a \\
  b
\end{pmatrix} \mapsto \begin{pmatrix}
  5 & 2 \\
  2 & 1
\end{pmatrix} \begin{pmatrix}
  a \\
  b
\end{pmatrix}
\]

The largest eigenvalue of the matrix is \( \lambda = 1 + \sqrt{2} \simeq 2.41 \). Hence, asymptotically, the length of the ‘blob’ is multiplied by 2.41 for each full stirring period.
Index formulas

To classify the possible train tracks for $n$ rods, we use two index formulas: these are standard and relate singularities to topological invariants, such as the Euler characteristic, $\chi$, of a surface.

Start with a sphere, which has $\chi = 2$. Each rod decreases $\chi$ by 1 (Euler–Poincaré formula), and the outer boundary counts as a rod. Thus, for our stirring device with $n$ rods, we have $\chi = 2 - (n + 1) = 1 - n$.

Now for the first index theorem: the maximum number of singularities in the foliation is $-2\chi = 2(n - 1)$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>max singularities</th>
<th>max bulk singularities</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>
Second index formula

\[ \sum_{\text{singularities}} \{2 - \#\text{prongs}\} = 2\chi(\text{sphere}) = 4 \]

where \#\text{prongs} is the number of prongs in each singularity (1-prong, 3-prong, etc).

Thus, each type of singularity gets a weight:

<table>
<thead>
<tr>
<th>#\text{prongs}</th>
<th>{2 - #\text{prongs}}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
| 2               | 0                        | hyperbolic point (\begin{tikzpicture}
  
  \draw (0,0) circle (0.1cm);
  \draw (-0.5,0) -- (0.5,0);

  \end{tikzpicture}) |
| 3               | -1                       |
| 4               | -2                       |
Each rod has a 1-prong singularity (⊙⃣). Hence, for 3 rods,

$$3 \cdot 1 + N = 4 \quad \implies \quad N = 1.$$ 

A 1-prong is the only way to have \(2 - \#\text{prongs} > 0\), hence there must be another one-prong! This corresponds to a boundary singularity (one injection point).

Our first index theorem says that there can be no other singularities in the foliation.

Kidney-shaped mixing regions are thus ubiquitous for 3 rods.
Counting singularities: 4 rods

For 4 rods,

\[ 4 \cdot 1 + N = 4 \quad \implies \quad N = 0. \]

Since every boundary component must have a singularity (part of the TN theorem), two cases:

1. A 2-prong singularity on the boundary \((N = 0)\), or
2. A 1-prong on the boundary and a 3-prong in the bulk \((N = 1 - 1 = 0)\).

Again, our first index formula says that we are limited to one bulk singularity.

\[ \implies \text{Two types of train tracks for } n = 4! \]
Two types of stirring protocols for 4 rods

2 injection points
Cannot be on same side

1 injection point
1 3-prong singularity
Five Rods, 3 Injection Points
Periodic Array of Rods

- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1 \sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).

$\chi = 1 + \sqrt{2}$ is the Silver Ratio!

- The entropy per ‘switch’ is $\log \chi$, where $\chi = 1 + \sqrt{2}$ is the Silver Ratio!
- This is optimal for a periodic lattice of two rods (Follows from D’Alessandro et al. (1999)).
Silver Mixers!

- The designs with entropy given by the silver ratio can be realised with simple gears.
- All the rods move at once: very efficient.
Four Rods

[movie 5] [movie 6] [movie 7]
Six Rods

[movie 8]
Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amount of entropy (stretching of material lines).
- Topology also predicts injection into the mixing region, important for open flows.
- Classify all rod motions according to their topological properties.
- More generally: Periodic orbits! (ghost rods and folding)
- We have an optimal design (silver mixers), but more can be done.
- Need to also optimise other mixing measures, such as variance decay rate.
- Three dimensions! (microfluidics)


