Experiment of Boyland et al.

$\sigma_1$ and $\sigma_2$ are referred to as the **generators** of the 3-braid group.
Two Stirring Protocols

$\sigma_1 \sigma_2$ protocol

$\sigma_1^{-1} \sigma_2$ protocol

Braiding

\( \sigma_1 \sigma_2 \) protocol

\( \sigma_1^{-1} \sigma_2 \) protocol

Let $I$ and $II$ denote the lengths of the two segments. After a $\sigma_2$ operation, we have

\[
\begin{pmatrix}
I' \\
II'
\end{pmatrix} = \begin{pmatrix}
I + II \\
II
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
I \\
II
\end{pmatrix} = \sigma_2 \begin{pmatrix}
I \\
II
\end{pmatrix}.
\]

Hence, the matrix representation for $\sigma_2$ is

\[
\sigma_2 = \begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
\]
Matrix Representation of $\sigma_{1}^{-1}$

Similarly, after a $\sigma_{1}^{-1}$ operation we have

$$
\begin{pmatrix}
I' \\
II'
\end{pmatrix} = \begin{pmatrix}
I \\
I + II
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
I \\
II
\end{pmatrix} = \sigma_{1}^{-1} \begin{pmatrix}
I \\
II
\end{pmatrix}.
$$

Hence, the matrix representation for $\sigma_{1}^{-1}$ is

$$
\sigma_{1}^{-1} = \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}.
$$
Matrix Representation of the Braid Group

We now invoke the faithfulness of the representation to complete the set,

\[
\sigma_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix};
\]

\[
\sigma_1^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}; \quad \sigma_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.
\]

Our two protocols have representation

\[
\sigma_1 \sigma_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}; \quad \sigma_1^{-1} \sigma_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.
\]
The Difference between the Protocols

- The matrix associated with each generator has unit eigenvalues.

The first stirring protocol has eigenvalues on the unit circle. The second has eigenvalues \((3 \sqrt{5})^{1/2} = 2^{1/2}\). So for the second protocol the length of the lines I and II grows exponentially! The larger eigenvalue is a lower bound on the growth factor of the length of material lines. That is, material lines have to stretch by at least a factor of \(2^{1/2}\) each time we execute the protocol.
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- The larger eigenvalue is a lower bound on the growth factor of the length of material lines. That is, material lines have to stretch by at least a factor of \(2^{\frac{3}{2}}\) each time we execute the protocol.
- This is guaranteed to hold in some neighbourhood of the rods (Thurston–Nielsen theorem).
The Difference between the Protocols

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- The first stirring protocol has eigenvalues on the unit circle.
- The second has eigenvalues \((3 \pm \sqrt{5})/2 = 2.6180\) for the larger eigenvalue.

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Freely-moving Rods in a Cavity Flow

[A. Vikhansky, Physics of Fluids 15, 1830 (2003)]
Particle Orbits are Topological Obstacles

Choose any fluid particle orbit (green dot).

Material lines must bend around the orbit: it acts just like a “rod”!

The idea: pick any three fluid particles and follow them.

How do they braid around each other?
In the second case there is no net braid: the two elements cancel each other.
We end up with a sequence of braids, with matrix representation

$$\sum^{(N)} = \sigma^{(N)} \ldots \sigma^{(2)} \sigma^{(1)}$$

where $\sigma^{(\mu)} \in \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ and $N$ is the number of braiding events detected after a time $t$. 
We end up with a sequence of braids, with matrix representation

$$\Sigma^{(N)} = \sigma^{(N)} \ldots \sigma^{(2)} \sigma^{(1)}$$

where $\sigma^{(\mu)} \in \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ and $N$ is the number of braiding events detected after a time $t$.

The largest eigenvalue of $\Sigma^{(N)}$ is a measure of the complexity of the braiding motion, called the braiding factor.

Random matrix theory says that the braiding factor can grow exponentially! We call the rate of exponential growth the braiding Lyapunov exponent or just braiding exponent.
Non-braiding Motion

First consider the motion of three points in concentric circles with irrationally-related frequencies.

The braiding factor grows linearly, which means that the braiding exponent is zero. Notice that the eigenvalue often returns to unity.
To demonstrate good braiding, we need a chaotic flow on a bounded domain (a spatially-periodic flow won’t do).

Aref’s blinking-vortex flow is ideal.

The only parameter is the circulation $\Gamma$ of the vortices.
Blinking Vortex: Non-braiding Motion

For $\Gamma = 0.5$, the blinking vortex has only small chaotic regions.

One of the orbits is chaotic, the other two are closed.
For $\Gamma = 13$, the blinking vortex is globally chaotic.

The braiding factor now grows exponentially. In the same time interval as for $\Gamma = 0.5$, the final value is now of order $10^{20}$ rather than 80!
Averaging over many Triplets

\[ t = 13 \]

\[ \Gamma = 13 \]

slope = 0.187

Averaged over 100 random triplets.
Comparison with Lyapunov Exponents

$\Gamma$ varies from 8 to 20.
Beyond Three Particles

\[ \Gamma = 16.5 \]
But does it Saturate?

Well, it really should...
One Rod Mixer: The Kenwood Chef

More than a Mixer at Christmas

and all through the year...

The Kenwood Chef

Your Saviour, Madam!

The Kenwood "Chef" not only...

- MIXES, KNEADS and Whips...
- with its wide range of oth-
rnal attachments, etc.
- LIQUEURS, Blends, Purées
  and Extracts JUICES
- SUGAR, Minced, Grinds, Shyres
- PIES, Potatoes and
- Even Opening Cans

Yes, you need the Kenwood
"Chef" in your home every day
of the year. It makes every meal
a festive occasion, with tempting
cookies and dishes saving the time
professional cooks.

What a wonderful Christmas gift... especially if it arrives in good time
to prepare the Christmas spread.

Other delightful Kenwood Gifts...

Standard Pack... £28 9s 6d

The Stainless Steel Stand... £30 10s

The KENWOOD ELECTRIC CO LTD, 25 Hoxton Square, London, W1
Topologically, the KC really shouldn’t mix well...
But the Good People at Kenwood Knew Better
Conclusions

• Topological chaos involves moving obstacles in a 2D flow, which create nontrivial braids.

• The complexity of a braid can be represented by the largest eigenvalue of a product of matrices—the braiding factor.

• Any collection of \( n \) particles can potentially braid.

• The complexity of the braid is a good measure of chaos.

• No need for infinitesimal separation of trajectories or derivatives of the velocity field.

• Many issues to investigate: faithfulness of representation, lower-bound for topological entropy…