Topological optimization and the simplest maps

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The taffy puller

[Photo and movie by M. D. Finn.]

[movie 1]
The mixograph

Model experiment for kneading bread dough:

[Department of Food Science, University of Wisconsin. Photos by J-LT.]
Planetary Mixers

In food processing, **rods** are often used for stirring.

[movie 2] ©BLT Inc.
Experiment of Boyland, Aref, & Stremler

Braid description of taffy puller

The three rods of the taffy puller in a space-time diagram. Defines a braid on \( n = 3 \) strands, \( \sigma_1^2 \sigma_2^{-2} \) (three periods shown).
Braid description of mixograph

\[ \sigma_3 \sigma_2 \sigma_3 \sigma_5 \sigma_6^{-1} \sigma_2 \sigma_3 \sigma_4 \sigma_3 \sigma_1^{-1} \sigma_2^{-1} \sigma_5 \]

braid on \( B_7 \), the braid group on 7 strands.
Topological entropy of a braid

**Burau representation** for 3-braids:

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$[\sigma_1^{-1} \sigma_2] = [\sigma_1^{-1}] \cdot [\sigma_2] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$}

This matrix has **spectral radius** $$(3 + \sqrt{5})/2$$ (Golden Ratio$^2$), and hence the topological entropy is $$\log[(3 + \sqrt{5})/2].$$

This is the growth rate of a ‘rubber band’ caught on the rods.

This matrix trick only works for 3-braids, unfortunately.
Optimizing over generators

- Entropy can grow without bound as the length of a braid increases;
- A proper definition of optimal entropy requires a cost associated with the braid.
- Divide the entropy by the smallest number of generators required to write the braid word.
- For example, the braid $\sigma_1^{-1} \sigma_2$ has entropy $\log[(3 + \sqrt{5})/2]$ and consists of two generators.
- Its Topological Entropy Per Generator (TEPG) is thus $\frac{1}{2} \log[(3 + \sqrt{5})/2] = \log[\text{Golden Ratio}].$
- Assume all the rods move.
Optimal braid

- In $B_3$ and $B_4$, the optimal TEPG is $\log[\text{Golden Ratio}]$.
- Realized by $\sigma_1^{-1}\sigma_2$ and $\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_2$, respectively.
- In $B_n$, $n > 4$, the optimal TEPG is $< \log[\text{Golden Ratio}]$.

Why? Recall Burau representation:

$$[\sigma_1] = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad [\sigma_2] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

Its spectral radius provides a lower bound on entropy. However,

$$|[\sigma_1]| = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad |[\sigma_2]| = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

provides an upper bound! Need to find Joint Spectral Radius.
Periodic Array of Rods

- Consider periodic lattice of rods.
- Move all the rods such that they execute $\sigma_1 \sigma_2^{-1}$ with their neighbor (Boyland et al., 2000).

![Diagram of periodic array of rods]

- The entropy per ‘switch’ is $\log(1 + \sqrt{2})$, the Silver Ratio!
- This is optimal for a periodic lattice of two rods (follows from D’Alessandro et al. (1999)).
- Also optimal if we assign cost by simultaneous operation.
Silver Mixers!

- The designs with entropy given by the Silver Ratio can be realized with simple gears.
- All the rods move at once: very efficient.
Build it!

[movie 6] [movie 7]
Experiment: Silver mixer with four rods
Silver mixer with six rods
The Minimizer problem

- On a given surface of genus $g$, which pA has the least $\lambda$?
- If the foliation is orientable (vector field), then things are much simpler;
- Action of the pA on first homology captures dilatation $\lambda$;
- Polynomials of degree $2g$;
- Procedure:
  - We have a guess for the minimizer;
  - Find all integer-coefficient, reciprocal polynomials that could have smaller largest root;
  - Show that they can’t correspond to pAs;
  - For the smallest one that can, construct pA.
Newton’s formulas

We need an efficient way to bound the number of polynomials with largest root smaller than $\lambda$. Given a reciprocal polynomial

$$P(x) = x^{2g} + a_1 x^{2g-1} + \ldots + a_2 x^2 + a_1 x + 1$$

we have Newton’s formulas for the traces,

$$\text{Tr}(\phi^k) = - \sum_{m=1}^{k-1} a_m \text{Tr}(\phi^{k-m}) - ka_k,$$

where

- $\phi$ is a (hypothetical) pA associated with $P(x)$;
- $\phi_*$ is the matrix giving the action of the pA $\phi$ on first homology;
- $\text{Tr}(\phi_*)$ is its trace.
Bounding the traces

The trace satisfies

\[ |\text{Tr}(\phi_k^*)| = \left| \sum_{m=1}^{g} (\lambda_m^k + \lambda_m^{-k}) \right| \leq g(r^k + r^{-k}) \]

where \( \lambda_m \) are the roots of \( \phi_* \), and \( r = \max_m(|\lambda_m|) \).

- Bound \( \text{Tr}(\phi_k^*) \) with \( r < \lambda, k = 1, \ldots, g \);
- Use these \( g \) traces and Newton’s formulas to construct candidate \( P(x) \);
- Overwhelming majority have fractional coeffs \( \rightarrow \) discard!
- Carefully check the remaining polynomials:
  - Is their largest root real?
  - Is it strictly greater than all the other roots?
  - Is it really less than \( \lambda \)?
- Largest tractable case: \( g = 8 \) \( 10^{12} \) polynomials.
Lefschetz’s fixed point theorem

This procedure still leaves a fair number of polynomials — though not enormous (10’s to 100’s, even for $g = 8$.)

The next step involves using Lefschetz’s fixed point theorem to eliminate more polynomials:

$$L(\phi) = 2 - \text{Tr}(\phi^*) = \sum_{p \in \text{Fix}(\phi)} \text{Ind}(\phi, p)$$

where

- $L(\phi)$ is the Lefschetz number;
- $\text{Fix}(\phi)$ is set of fixed points of $\phi$;
- $\text{Ind}(\phi, p)$ is index of $\phi$ at $p$.

We can easily compute $L(\phi^k)$ for every iterate using Newton’s formula.
Eliminating polynomials

Outline of procedure: for a surface of genus $g$,

- Use the Euler–Poincaré formula to list possible singularity data for the foliations;
- For each singularity data, compute possible contributions to the index (depending on how the singularities and their separatrices are permuted);
- Check if index is consistent with Lefschetz’s theorem.

With this, we can reduce the number of polynomials to one or two!
Minimizers for orientable foliations

<table>
<thead>
<tr>
<th>$g$</th>
<th>polynomial</th>
<th>minimizer</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$X^4 - X^3 - X^2 - X + 1$</td>
<td>$\approx 1.72208$ †</td>
</tr>
<tr>
<td>3</td>
<td>$X^6 - X^4 - X^3 - X^2 + 1$</td>
<td>$\approx 1.40127$</td>
</tr>
<tr>
<td>4</td>
<td>$X^8 - X^5 - X^4 - X^3 + 1$</td>
<td>$\approx 1.28064$</td>
</tr>
<tr>
<td>5</td>
<td>$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$</td>
<td>$\approx 1.17628$ *</td>
</tr>
<tr>
<td>6</td>
<td>$X^{12} - X^7 - X^6 - X^5 + 1$</td>
<td>$\succeq 1.17628$</td>
</tr>
<tr>
<td>7</td>
<td>$X^{14} + X^{13} - X^9 - X^8 - X^7 - X^6 - X^5 + X + 1$</td>
<td>$\approx 1.11548$</td>
</tr>
<tr>
<td>8</td>
<td>$X^{16} - X^9 - X^8 - X^7 + 1$</td>
<td>$\approx 1.12876$</td>
</tr>
</tbody>
</table>

† Zhirov (1995)’s result; also for nonorientable [Lanneau–T];
* Lehmer’s number; realized by Leininger (2004)’s pA;
• For genus 6 we have not explicitly constructed the pA;
• Genus 6 is the first nondecreasing case.
• Genus 7 and 8: pA’s found by Aaber & Dunfield (2010) and Kin & Takasawa (2010b) [$g = 7$]; Hironaka (2009) [$g = 8$].
Conclusions

- Having rods undergo ‘braiding’ motion guarantees a minimal amount of entropy (stretching of material lines).
- Can optimize to find the best rod motions, but depends on choice of ‘cost function.’
- For two natural cost functions, the Golden Ratio and Silver Ratio pop up!
- Found orientable minimizer on surfaces of genus $g \leq 8$; only known nonorientable case is for genus 2.