Exact topological entropy for some non-hyperbolic maps

Jean-Luc Thiffeault\textsuperscript{1,2} Sarah Matz\textsuperscript{1} Kevin Mitchell\textsuperscript{3}
Special guest: Phil Boyland\textsuperscript{4}

\textsuperscript{1}Department of Mathematics  
University of Wisconsin – Madison

\textsuperscript{2}Institute for Mathematics and its Applications  
University of Minnesota – Twin Cities

\textsuperscript{3}University of California – Merced

\textsuperscript{4}Department of Mathematics  
University of Florida

AMS Spring Central Section Meeting, St-Paul, 10 April 2010
Stirring with rods

When stirring a viscous fluid with rods, a blob of dye gets ‘caught’ on the rods.

The rod motion can be connected to the isotopy class of the induced map [Boyland et al. (2000); Thiffeault & Finn (2006)]. [movie 1]
Describing the rod motion

Express in terms of generators of the braid group:

- $\sigma_1$ is the clockwise interchange of the first and second rods;
- $\sigma_2$ is the clockwise interchange of the second and third rods.

Any stirring protocol (rod motion) can be represented as a combination of these generators. We consider protocols of the form $\sigma_1^k \sigma_2^{-\ell}$.

Two types:

- counter-rotating ($k\ell > 0$);
- co-rotating ($k\ell < 0$).

The protocol on the previous slide had $k = \ell = 1$: 
Action on homology

Find the growth rate of material lines — topological entropy.

Can use Burau matrix representation:

\[
\begin{align*}
[\sigma_1] &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \\
[\sigma_2] &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}
\end{align*}
\]

\[
[\sigma_1^k \sigma_2^{-\ell}] = \begin{pmatrix} 1 + k\ell & k \\ \ell & 1 \end{pmatrix}
\]

These matrices tell us about how loops are transformed.

\([\sigma_1^k \sigma_2^{-\ell}] \) is a hyperbolic matrix (|largest eigenvalue| > 1) if

\[|2 + k\ell| > 2\]
If $[\sigma_1^k \sigma_2^{-\ell}]$ is hyperbolic, then the protocol is isotopic to a pseudo-Anosov mapping with positive entropy.

For instance, if $k = \ell = 1$ (counter-rotating) or $k = 1$, $\ell = -5$ (co-rotating), then $|2 + k\ell| = 3$, and

$$h = \log |\text{largest eigenvalue}| = \log(\frac{1}{2}(3 + \sqrt{5}))$$

The only difference is that in the counter-rotating case the eigenvalue of the matrix $[\sigma_1^k \sigma_2^{-\ell}]$ is positive, while for the co-rotating case it is negative.

This $h$ is a lower bound on the growth of material lines in the flow.
Stretching of material lines

$\sigma_1 \sigma_2^{-1}$

$\sigma_1 \sigma_2^5$
Huge gap between lower bound and measured rate for the co-rotating case. Why?
Consider a simpler problem: ‘Dehn twists’ on a strip on the torus.

Compose these two maps together: Linked Toral Twist Map.

[Devaney, Przytycki, Burton & Easton, ... see Sturman et al. (2006).]
Linked toral twist maps (LTTMs)

Regions: fixed (gray), one map (light blue), both maps (dark blue).

\[ \alpha = \beta = 1: \text{recover Anosov homeomorphism (same isotopy class).} \]
Line growth for LTTMs

counter-rot. \((k\ell = 1 > 0)\)
entropy \(\simeq .962\)
(same as Anosov class)

co-rot. \((k\ell = -5 < 0)\)
entropy \(\simeq 1.91 > .962\)
Line growth for LTTMs: Unstable manifold

Unstable manifold on the universal cover:

counter-rot. \((k\ell = 1 > 0)\)

co-rot. \((k\ell = -5 < 0)\)
Extra growth comes from ‘folds’
Spine

For small $\alpha, \beta$, squish strip on ‘spine’ (co-rotating, $k \ell < 0$):

Unstable manifold is then a ‘staircase,’ with some defects.
Spine: entropy

On the spine, can see the extra entropy comes from material lines not being ‘pulled tight’.

⇒ when looking at the action of the map on words in $\pi_1$ (loops), we shouldn’t cancel loops with their inverses (leave them dangling).

This is the same as taking absolute values for the action on $\pi_1$:

$$\begin{pmatrix} 1 + |k\ell| & |k| \\ |\ell| & 1 \end{pmatrix}$$

For $k = 1$, $\ell = -5$, this gives an entropy of 1.92. Compare to the numerically-computed value 1.98 (as $\alpha, \beta \to 0$).

So there’s a bit more entropy, but close! Not clear yet where the extra growth comes from...


