Lecture 34: Permutations generated by Brownian particles

1 Brownian particles on the real line

Consider \( n \) Brownian particles on the real line, with diffusion constant \( D \). The position of each particle is denoted \( x_k(t) \). The position of a walker at time \( t \) has a probability density function \( p(x, t; x', 0) \) that satisfies the heat equation,

\[
\frac{\partial p}{\partial t} - D \Delta p = \delta(x - x') \delta(t),
\]

where \( x' \) is the initial position of the particle, with solution

\[
p(x, t; x', 0) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-x')^2}{4Dt}}, \quad t > 0.
\]

Assume the initial ordering

\[
x_1(0) < x_2(0) < \cdots < x_n(0),
\]

and define the probability \( P(t, s) \) that the particles are ordered according to the permutation \( s \in S_n \), the symmetric group on \( n \) symbols; thus,

\[
P(0, s) = \begin{cases} 1, & s = \text{id}; \\ 0, & \text{otherwise}. \end{cases}
\]

At later times, we have

\[
P(t, s^{-1}) = \int_{-\infty}^{\infty} dx_{s(1)} \int_{x_{s(1)}}^{\infty} dx_{s(2)} \cdots \int_{x_{s(n-1)}}^{\infty} dx_{s(n)} \prod_{k=1}^{n} p(x_{s(k)}, t; x_{s(k)}(0), 0).
\]

That is, the particle \( s(1) \) is to the left of all the others, \( s(2) \) is to the right of \( s(1) \) but to the left of the rest, etc.
Now let
\[ y_k := x_{s(k)}/\sqrt{4Dt}, \quad \varepsilon_k(t) := x_{s(k)}(0)/\sqrt{4Dt}, \]
from which (5) becomes
\[ P(t, s^{-1}) = \int_{-\infty}^{\infty} dy_1 \int_{y_1}^{\infty} dy_2 \cdots \int_{y_{n-1}}^{\infty} dy_n \prod_{k=1}^{n} \frac{1}{\sqrt{\pi}} e^{-\left(y_k - \varepsilon_k(t)\right)^2}. \] (7)

For large time, we have \( \varepsilon_k(t) \ll 1 \), and
\[ e^{-(y_k - \varepsilon_k(t))^2} = e^{-y_k^2} (1 + 2\varepsilon_k y_k) + O(\varepsilon^2). \] (8)

We must have
\[ \frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} dy_1 \int_{y_1}^{\infty} dy_2 \cdots \int_{y_{n-1}}^{\infty} dy_n e^{-\sum_{k=1}^{n} y_k^2} = \frac{1}{n!}. \] (9)
(This must work for any PDF with the right symmetry, so must be the fraction of volume occupied by an \( n \)-dimensional ‘wedge.’)

Define
\[ I_{n,\ell} := -\frac{1}{\pi^{n/2}} \int_{-\infty}^{\infty} dy_1 \int_{y_1}^{\infty} dy_2 \cdots \int_{y_{n-1}}^{\infty} dy_n y_{\ell} e^{-\sum_{k=1}^{n} y_k^2}. \] (10)

By changing order of integration and replacing \( y_k \) by \( -y_{n-k+1} \), we can show \( I_{n,\ell} = -I_{n,n-\ell+1} \). Some specific values are \( I_{2,1} = -I_{2,2} = 1/(2\sqrt{2\pi}) \), \( I_{3,1} = -I_{3,3} = 1/(4\sqrt{2\pi}) \), \( I_{3,2} = 0 \). Challenge: compute this in general (must be known...).

In any case, the time-asymptotic solution is
\[ P(t, s^{-1}) = \frac{1}{n!} - \sum_{k=1}^{n} 2\varepsilon_k I_{n,k} + O(\varepsilon^2), \] (11)
which, from (6), shows a rather slow approach to the uniform distribution as \( 1/\sqrt{t} \).

2 Brownian particles on the unit interval

Now we turn to Brownian particles on the interval \([0, 1]\), with reflecting boundary conditions at the endpoints (Figure 1). The same heat equation (2) is satisfied by the
probability density (Green’s function), but now the reflecting (Neumann) boundary conditions lead to

\[ p(x, t; x', 0) = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{4\pi Dt}} \left( e^{-\frac{(x-x'-2k)^2}{4Dt}} + e^{-\frac{(x+x'-2k)^2}{4Dt}} \right), \quad t > 0, \]  

(12)

which can also be written

\[ p(x, t; x', 0) = \sum_{k=-\infty}^{\infty} \cos(\pi k x) \cos(\pi k x') e^{-\frac{(\pi k)^2}{2} Dt}, \quad t > 0. \]  

(13)

The same formula (5) applies for \( P(t, s^{-1}) \). Let’s take \( n = 2 \); then after some integrals

\[ P(t, s^{-1}) = \frac{1}{2} + \sum_{k \neq \ell} \frac{(1 - (-1)^{k+\ell})}{\pi^2 (k^2 - \ell^2)} \cos(\pi k x_n(1)(0)) \cos(\pi k x_n(2)(0)) e^{-\pi^2 (k^2 + \ell^2) Dt}. \]  

(14)

(\( \sum' \) means \( k \neq 0 \) and \( \ell \neq 0 \).) The slowest exponential has \( k^2 + \ell^2 = 1 \); hence, we have

\[ \left| P(t, s^{-1}) - \frac{1}{2} \right| \leq |C| e^{-\pi^2 Dt} \]  

(15)

where

\[ C = \sum' \left| \frac{(1 - (-1)^{k+\ell})}{\pi^2 (k^2 - \ell^2)} \right|, \]  

(16)

as long as \( C \) is finite. (Challenge: does this diverge? If so need to refine the analysis.)

In any case it appears to be the right bound: see Fig. 2. Is this a cut-off? How do we show this?

Figure 1: Five Brownian particles on the interval \([0, 1]\), with reflecting boundary conditions.
Figure 2: Variation distance as a function of time for 2 particles, with $D = 5 \times 10^{-6}$ (100,000 realizations). The dashed line is proportional to $e^{-\pi^2Dt}$. 