Lecture 20: Topological mixing on the torus

Stirring by moving rods \([\text{movie}]\) \{ fluids (viscous) \}
\{ elastic bodies (bread, taffy) \}

Repeat: line length grows exponentially in this case.

How do we characterize this? A lot of insight obtained from first considering the torus.

Homeomorphism \( \mathbb{T}^2 \rightarrow \mathbb{T}^2 \) invertible, continuous with continuous inverse.

\( \text{Homeo}^+ \) \( \mathbb{T}^2 \). \( \text{Homeo}^+ \) \( \mathbb{T}^2 \) is a group under composition of functions.
Define: \[ \text{MCG}(T^2) = \text{Homeo}^+(T^2) / \text{isotopy} \]

(mapping class group of \( T^2 \))

(inherits the group structure of \( \text{Homeo}^+(T^2) \))

What does \( \text{MCG}(T^2) \) look like?

Consider an induced homomorphism on \( \pi_1(T^2, x_0) \)

Fundamental group of \( T^2 \) with basepoint \( x_0 \)

\[ f: T^2 \to T^2, \quad f_*: \pi_1(T^2) \to \pi_1(T^2) \]

Hence, \( f_* \) given by matrices

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z} \quad \text{and} \quad \det(a, b; c, d) = ad - bc \neq 0 \]

But also \( f \circ f^{-1} = \text{id} \) \( \Rightarrow f_* \circ f_*^{-1} = I \) so \( f_* \) invertible

Let \( m = ad - bc \neq 0 \). We have also \( f_*^{-1}: \mathbb{Z}^2 \to \mathbb{Z}^2 \) so

\[ f_*^{-1} = \frac{1}{m} \begin{pmatrix} d & -b \\ c & a \end{pmatrix}, \quad \text{so need all entries} \in \mathbb{Z}. \]

4. \( m \) divides every entry

Let \( a = ma, \ b = mb, \ c = mc, \ d = md, \ \alpha, \beta, \gamma, \delta \) integers.
Then \( m = ad - bc = m^2 (\alpha \delta - \beta \gamma) \Rightarrow 1 = m (\alpha \delta - \beta \gamma) \).

Since all integers need \( m = \pm 1 \). \( m = \pm 1 \Rightarrow \text{orientable} \).

Hence, \( \text{MCG}(T^2) = \text{SL}(2,\mathbb{Z}) \). Why is this?

Now, how do we classify the elements of this group?

Look at eigenvalues. \( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} - x I = x^2 - (a+d)x + ad-bc \)

Let \( \tau = a+d \) (trace) \[ \text{Note: } p(M) = M^2 - \tau M + I = 0 \]

Characteristic polynomial: \( p(x) = x^2 - \tau x + 1 \)

Eigenvalues: \( x = \frac{1}{2} (\tau \pm \sqrt{\tau^2 - 4}) \) So \(|\tau| = 2 \) important.

Let's examine different cases.

1) \(|\tau| < 2 \). \( \tau = -1, 0, 1 \).

If \( \tau = 0 \), then \( p(M) = M^2 + I = 0 \Rightarrow M^2 = -I \Rightarrow \boxed{M^4 = I} \)
If \( \alpha = \pm 1 \), \( p(M) = M^2 + M + \mathbf{I} \Rightarrow M^2 = \pm M - \mathbf{I} \)
\[
M^3 = M (\pm M - \mathbf{I})
\]

Either way, we can write
\[
M^{12} = \mathbf{I}, \quad |\alpha| < 2
\]

This is called finite order. After applying \( f \) enough times, it is isomorphic to the identity map.

2) \(|\alpha| = 2\): Then eigenvalues are both \( \pm 1 \) (\( = \frac{2}{1} \))

\( M^2 + 2M + \mathbf{I} = (M + \mathbf{I})^2 = 0 \Rightarrow M = \pm \mathbf{I} + N \), \( N^2 = 0 \)

\[
\begin{pmatrix}
\alpha & c \\
c & d
\end{pmatrix} = \begin{pmatrix}
\frac{a}{2} & c \\
c & d
\end{pmatrix} = \begin{pmatrix}
\frac{a}{2} + bc & c(\frac{a}{2} + cd) + ad \\
c(\frac{a}{2} + cd) + cd & d(\frac{a}{2} + cd)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{a}{2} & c \\
c & \frac{a}{2}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\alpha - \frac{a}{2} \\
c \frac{a}{2}
\end{pmatrix}
\]

\( \sin \frac{\pi}{2} = \frac{\pi}{2} \) (\( \mathbf{C} \) = \( \pm 1 \))

Hence, the homotopy classes given by \( \begin{pmatrix}
\alpha & c \\
c \frac{a}{2}
\end{pmatrix} \) are invariant (or reverse direction) under \( M \).

\( \Rightarrow \) invariant curve (called reducible)
Let \( R = \begin{pmatrix} 1 & a - \tau_2 \\ 0 & c \end{pmatrix} \). Then:

\[
R^{-1}MR = \begin{pmatrix} \tau_2 & 0 \\ 1 & \tau_2 \end{pmatrix}
\]

Jordan form

Simplest type: \( M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), so \( M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), \( M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

Next time: case 3) \( |\tau| > 2 \)!