Lecture 13: Rate of decay and local stretching

Recall from many lectures ago that for a 2D, extensional, incompressible flow, the concentration of a passive scalar:

$$\theta(x,y,t) \sim e^{-\lambda t}$$

with a Gaussian cross-section.

Now imagine the flow is being subjected to a random renewing flow.

Assume that \( \tau \) (the correlation time) is large enough that our Gaussian blob aligns rapidly:

$$\lambda \gg 1$$

Then at each application the intensity of \( \theta \) decays by a factor of

$$e^{-\lambda \tau}$$

Consider \( \langle |\theta|^p \rangle \) when the expected value is over the random matrices in our flow.
Then:

$$<|\Theta|^p> \sim \int e^{-\rho h} e^{-Nh} dh$$

$$\sim \int e^{-N(\rho h + g(h))} dh$$

For large $N$, the rate of decay is thus

$$\gamma(p) = \frac{1}{2} \inf_{h} (\rho h + g(h))$$

$$<|\Theta|^p> \sim e^{-\gamma(p) t}$$

$$t \to \infty$$

Thus, $\gamma(p) = -\ell(-p)$

[generalized Lyapunov exponent]
There is one crucial difference: we cannot consider negative $h$. Why? Here is the stretching of the longest axis of the ellipsoid:

This is a nondecreasing quantity, unlike the growth of a line segment $l$. 
So consider: 

\[ \tau(p) = \inf_{h>0} \left( ph + g(h) \right) \]

Convex saddle point

\[ p + g'(h(p)) = 0 \]

\[ \tau(p) = ph^*_x(p) + g(h^*_x(p)) \]

\[ \tau'(p) = h^*_x(p) + g'(h^*_x(p)) h^*_x(p) \]

\[ + ph^*_x(p) \]

\[ = h^*_x(p) \]

So \( \tau'(p) = 0 \) when \( h^*_x(p) = 0 \Rightarrow h = 0 \).

So the point were the saddle point is \( h = 0 \)

coincides with the extremum of \( \lambda(p) \)!
We must amend our figure for $\tau(p)$ to reflect this:

\[ \tau(p) = \begin{cases} \inf \left( ph + g(l) \right), & h_*(p) > 0 \\ g(0), & h_*(p) < 0 \end{cases} \]

Thus, $\tau(p)$ changes depending on the value of $h_*(p)$.

The decay of $\langle 1|\Omega|^p \rangle$ for large $p$ is thus completely dominated by realization with zero stretching.

$\rightarrow$ becomes independent of $p$. 