1. Let \( A \in \mathbb{C}^{m \times m} \) be arbitrary. The set of all Rayleigh quotients of \( A \), corresponding to all nonzero vectors \( x \in \mathbb{C}^m \), is known as the field of values or the numerical range of \( A \), a subset of the complex plane denoted by \( W(A) \).

(a) Show that \( W(A) \) contains the convex hull of the eigenvalues of \( A \).

(b) Show that if \( A \) is normal, then \( W(A) \) is equal to the convex hull of the eigenvalues of \( A \).

2. Let \( A \) be an \( n \times n \) tridiagonal matrix with \( a_{i,i+1} = a_{i+1,i} = -1 \) and \( a_{ii} = 3 \), and let \( b \in \mathbb{R}^n \). For which values of the parameter \( \omega \) does the iteration
\[
x_{k+1} = x_k + \omega (b - Ax_k), \quad k = 0, 1, 2, \ldots
\] converge to a solution of \( Ax = b \) for any starting value \( x_0 \in \mathbb{R}^n \)? Test your result on a computer for \( n = 5 \) and comment on your findings.

3. Consider the real system of linear equations \( Ax = b \), where \( A \) is nonsingular and satisfies \( (x, Ax) > 0 \) for all real \( x \neq 0 \), where \( (x, y) = x^T y \) is the Euclidean inner product.

(a) Show that \( (x, Ax) = (x, Mx) \) for all real \( x \), where \( M = (A + A^T)/2 \) is the symmetric part of \( A \).

(b) Prove that \( (x, Ax)/\|x\| \geq \lambda_{\text{min}}(M) > 0 \), where \( \lambda_{\text{min}}(M) \) is the smallest eigenvalue of \( M \).

(c) Consider the iterative sequence \( x_{n+1} = x_n + \alpha_n r_n \), where \( r_n = b - Ax_n \) is the residual, and \( \alpha_n \) is chosen to minimize \( \|r_{n+1}\|_2 \) as a function of \( \alpha_n \). Prove that
\[
\frac{\|r_{n+1}\|_2}{\|r_n\|_2} \leq \left( 1 - \frac{\lambda_{\text{min}}(M)^2}{\lambda_{\text{max}}(A^T A)} \right)^{1/2}.
\]

4. Let \( A \) be the 100 \( \times \) 100 tridiagonal symmetric matrix with 1, 2, ..., 100 on the diagonal and 1 on the sub- and super- diagonals, and set \( b = (1, 1, \ldots, 1)^T \). Write a program that takes 100 steps of the conjugate gradient algorithm, and separately a program that takes 100 steps of the steepest descent algorithm, to approximately solve \( Ax = b \). Produce a plot with four curves: the computed residual norms \( \|r_n\|_2 \) and the actual residual norms \( \|b - Ax_n\|_2 \) for CG, the residual norms \( \|r_n\|_2 \) for steepest descent, and the estimate \( 2(\sqrt{\kappa} - 1)^n/(\sqrt{\kappa} + 1)^n \). Comment on your results.

5. Prove that if \( w \) is continuous on \([0, 1] \), and
\[
\int_0^1 wv \, dx = 0 \quad \forall v \in V, \quad (3)
\]
\[
V = \{ v \in C[0, 1], v_x \text{ piecewise continuous}, v(0) = v(1) = 0 \}, \quad (4)
\]
then \( w(x) = 0 \) for \( x \in [0, 1] \).