Lecture 25: Dispersion

Beyond second order: 3rd order dispersion

\[ u_t + u_{xxx} = 0, \quad u(x,0) = e^{-x^2} \quad x \in \mathbb{R} \]

Fourier transform solution: \( \hat{u}(\xi, \kappa) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t,x) e^{-i\kappa x} \, dx \)

\[ \hat{u}_t + (i\hbar)^3 \hat{u} = 0 \]
\[ \hat{\xi} - i\kappa \hbar^3 \hat{\kappa} = 0 \]
\[ \hat{u} = \hat{f}(\kappa) e^{i\kappa^2 t} \]

\[ u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\kappa) e^{i(\hbar x + \hbar^2 t)} \, d\kappa \]

For example, for \( f(x) = e^{-x^2} \), \( \hat{f}(\kappa) = \frac{e^{-\kappa^2/4}}{\sqrt{2\pi}} \)

\[ u(t,x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\kappa^2/4} e^{i(\hbar x + \hbar^2 t)} \, d\kappa \]

\[ = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\kappa^2/4} \cos(\hbar x + \hbar^2 t) \, d\kappa \]
Rapid oscillations on left propagate away and spread.

Fundamental solution: \( f(x) = \delta(x) \), \( \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \)

\[ u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(hx + h^3t)} dh = \frac{1}{\pi} \int_{0}^{\infty} \cos(hx + h^3t) dh \]

Convergence?!

Note that

\[ \int_{0}^{l} \frac{\cos(hx + h^3t)}{x + \frac{h^3}{3}} dh = \int_{0}^{l} \frac{1}{x + \frac{h^3}{3}} \frac{d}{dh} \sin(hx + h^3t) dh \]

\[ = \frac{\sin(hx + h^3t)}{x + \frac{h^3}{3}} \bigg|_{0}^{l} + \int_{0}^{l} \frac{6h t \sin(hx + h^3t)}{(x + \frac{h^3}{3})^2} dh \]

\( \Rightarrow \) as \( l \to \infty \)

So the rapid oscillations make the integral convergent.
$u(t, x)$ not given analytically, but in terms of

$$Ai(z) = \frac{1}{\pi} \int_0^\infty \cos(sz + \frac{1}{3}z^3) \, ds \quad \text{Airy function}$$

$$u(t, x) = \frac{1}{(3t)^{1/3}} \, \text{Ai} \left( \frac{x}{(3t)^{1/3}} \right) \sim t^{-1/3}$$

Converge weakly to a $\delta$ function as $t \to 0$.

$$F(t, x; \xi) = \frac{1}{(3t)^{1/3}} \, \text{Ai} \left( \frac{x - \xi}{(3t)^{1/3}} \right)$$

Need better intuition... so back to start $u_t + u_{xxx} = 0$

Linear, cast coeffs, so look for solutions $e^{i(kx - \omega t)}$

$$\omega = -k^3 \quad \text{dispersion relation}$$

$$u(t, x) = e^{i(kx - \omega t)} = e^{i[k(x - c_\omega t)]}, \quad c_\omega = \frac{\omega}{k} \quad \text{phase velocity}$$
For \( w = ck \), \( c \) const., \( U_t + c U_x = 0 \)

Transport, no diffusion

In general,

\[
U(t, x) = \int_{-\infty}^{\infty} e^{i(hx - wt)} f(h) \, dh
\]

\( w = w(h) \)

Moving frame:

\[
U(t, x + ct) = \int_{-\infty}^{\infty} e^{i \varphi(h) + h x} f(h) \, dh = H(t)
\]

where \( \varphi(h) = ch - w(h) \), \( h(h) = e^{i\frac{h^2}{2}} \)

Oscillatory integral \( \to 0 \) as \( t \to \infty \).

Dominant behavior for large \( t \)? Stationary phase

\( \varphi' = 0 \)

\( \varphi'(h) = w'(h) - c \quad c_g = w'(h) \)

\( = 0 \)

For \( w = h^3 \), we have \( c_p = \frac{w}{h} = -h^2 \)

\( c_g = w' = -3h^2 \) fastest