Lecture 12: Separation of wave equation

Back to wave eqn: \( u_{tt} = c^2 u_{xx} \)

Separation of variables Ansatz: \( u(t,x) = \psi(t) \phi(x) \)

Plug in, separate: \( \frac{\psi''}{\psi} = \frac{c^2 \phi''}{\phi} = \lambda = \text{const} \)

\( \psi'' - \lambda \psi = 0, \quad \phi'' - \frac{\lambda}{c^2} \phi = 0 \)

Solve as before:
- \( \lambda > 0 \) \( \sin, \cos \)
- \( \lambda < 0 \) \( e^{\pm} \) or \( \sinh, \cosh \)
- \( \lambda = 0 \) \( 1, x \)

Assume solution is bounded in \( \mathbb{R}^2 \): \( 0 < \lambda = \omega^2 \)

\( u(t,x) = \sum_n (A_n \cos \omega_n t + B_n \sin \omega_n t)(C_n \cos \left( \frac{\omega_n x}{c} \right) + D_n \sin \left( \frac{\omega_n x}{c} \right)) \)

\( \cos a \cos b = \frac{1}{2} (\cos (a+b) + \cos (a-b)) \)

\( \sin a \sin b = \frac{1}{2} (\cos (a+b) - \cos (a-b)) \)

\( \cos a \sin b = \frac{1}{2} (\sin (a+b) - \sin (a-b)) \)

\( \sin a \cos b = \frac{1}{2} (\sin (a+b) + \sin (a-b)) \)
By using these we have
\[ u(t,x) = \sum_n \left[ A_n^{(r)} \cos \left( \frac{\omega_n (x - ct)}{c} \right) + B_n^{(r)} \sin \left( \frac{\omega_n (x - ct)}{c} \right) \right] \\
+ \sum_n \left[ A_n^{(l)} \cos \left( \frac{\omega_n (x + ct)}{c} \right) + B_n^{(l)} \sin \left( \frac{\omega_n (x + ct)}{c} \right) \right] \]

Recall the earlier general solution: \( u(t,x) = p(x-ct) + g(x+ct) \)

This is the same but in Fourier form. (Technically \( p, g \) should be periodic, but can be period \( \to \) and we use Fourier transform)

Recall now d'Alembert's formula:
\[ u(t,x) = \frac{1}{2} \left( f(x-ct) + f(x+ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(z) \, dz \]

Solves \( u_{tt} = c^2 u_{xx}, \ u(0,x) = f(x), \ u_t(0,x) = g(x) \).

To solve Dirichlet problem \( u(t,0) = u(t,l) = 0 \) \( 0 < x < l \) we either Fourier sine or cosine (no in \( x \), \( \omega_n \) odd periodic extension):
\[ \tilde{f}(x) = -\tilde{f}(-x), \quad \tilde{g}(x) = -\tilde{g}(-x) \]
\[ \tilde{f}(x+2l) = \tilde{f}(x), \quad \tilde{g}(x+2l) = \tilde{g}(x) \]
Then
\[ u(t,0) = \frac{1}{2} (\tilde{f}(-ct) + \tilde{f}(ct)) + \frac{1}{2c} \int_{-ct}^{0} \tilde{g}(z) \, dz = 0 \]

\[ u(t,l) = \frac{1}{2} (\tilde{f}(-l - ct) + \tilde{f}(l + ct)) + \frac{1}{2c} \int_{l - ct}^{l + ct} \tilde{g}(z) \, dz \]

\[ = \frac{1}{2} (\tilde{f}(-l - ct) + \tilde{f}(l + ct)) + \frac{1}{2c} \int_{-ct}^{ct} \tilde{g}(z + l) \, dz \]

\[ = 0 \]

\[ \tilde{g}(x + l) = \tilde{g}(x - l) = -\tilde{g}(-x + l) \]

So we use the full d'Alambert solution with \( \tilde{f}, \tilde{g} \), but only evaluate the result on \( 0 < x < l \). "Window" on periodic wave. "Interference" picture.

Similar trick for Neumann (\( u_x = 0 \)) condition using even periodic extension.

Q: can this work for \( u(t,0) = 0, \ u_x(t,l) = 0 \)?