Lecture 11: More on heat equation

A quick look at inhomogeneous B.C.:

\[ \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad u(t_0) = \alpha, \quad u(t, l) = \beta \]

The general method (for time-independent B.C.) is to solve the steady problem: (equilibrium)

\[ 0 = \alpha \frac{d^2 u}{dx^2}, \quad u(0) = \alpha, \quad U(l) = \beta \]

\[ \Rightarrow U(x) = \alpha + \frac{\beta - \alpha}{l} x \]

Then define \( \tilde{u}(t, x) = u(t, x) - U(x) \).

Easy to see that \( \tilde{u} \) has homogeneous B.C.'s.

Hence, can re-use homogeneous sol'n from last time:

\[ u(t, x) = \alpha + \frac{\beta - \alpha}{l} x + \sum_{n=1}^{\infty} b_n e^{-\frac{n^2 \pi^2 \alpha t}{l^2}} \sin \left( \frac{n \pi x}{l} \right) \]

(\( \sum \) can substitute Fourier series to equate coeffs with \( f(x) \)).

When B.C.s are time-dep., find particular solution instead. Get "source term" in \( \tilde{u} \) equation.
Robin BCs: Thijs get hairier with more general BCs:

\[ u_t = u_{xx}, \quad u(1,0) = 0, \quad u_x(1,1) + \beta u(1,1) = 0. \]

\( \beta = 0 \) is easy, so assume \( \beta \neq 0 \).

\[ u = e^{-\lambda t} N(x), \text{ etc... (separation of variables again)} \]

\[ N'' + \lambda N = 0, \quad N(0) = 0, \quad N'(1) + \beta N(1) = 0. \]

Again let \( w = \sqrt{\lambda} \).

\[ N(x) = a \cos wx + b \sin wx \]

\[ N'(1) + \beta N(1) = w \cos w + \beta \sin w = 0. \]

Transcendental eqn: \( \beta w = -\tan w \)

\( w = 0 \) trivial solution.

\[ w_n = \frac{(2n-1)\pi}{2} \quad \text{as} \ n \to \infty. \]

Also work for \( \beta < 0 \).
Here \( \lambda > 0 \) so there are exp. decaying solutions

\[ u_n(t, x) = e^{-\lambda t} \sin(w_n x), \quad w_n = \sqrt{\lambda} \]

\( \lambda = 0 \) is a special case: \( u(t) = e^{tx} \)

\[ u'(1) + \beta u(1) = 1 + \beta = 0 \quad \Rightarrow \quad \beta = -1 \]

Steady solution only exists if fluxes are bounded.

Now suppose \( \lambda = -w^2 = 0 \) \( \Rightarrow \) \( u(x) = \sinh w x \)

\[ u'(1) + \beta u(1) = w \cosh w + \beta \sinh w = 0 \]

\[ -\beta^{-1} w = \tanh w \]

Get trivial solution unless \( 0 < -\beta^{-1} < 1 \) or \( -\infty < \beta < -1 \)

Hence, get at most one eigensolution with \( \lambda < 0 \) when \( \beta < -1 \).

All other eigensolutions have \( \lambda > 0 \). (Decaying)
\[ u_0(t,x) = e^{\lambda_0 t} \sinh \omega_0 x \] \text{Growing eigenmode!}

This corresponds to a net heat flux into the system.

Another type of problem: grand heating

\[ u_t = \gamma u_{xx}, \quad x \geq 0, \quad u(t,0) = a \cos \omega t \]

\[-\infty < t < \infty \quad |u(t,x)| \leq M \quad \text{bounded} \]

\[ u(t,\infty) = 0 \]

\[ \text{Bounded in time mean} \]

\[ u(t,x) = e^{i\omega t} \sigma(t) \]

\[ \sigma N'' = i\omega \sigma \]

\[ x \quad \text{Re} N(0) = a, \quad N(\infty) = 0. \]

Put \( N(x) = e^{\alpha x} \) such that \( \alpha^2 = i\omega \)

\[ \alpha = \pm \sqrt{i\omega/\gamma} = \pm (1+i)\sqrt{\omega/2\gamma} \]

Take "+" to satisfy \( N(\infty) = 0. \)
\[ N(x) = N_0 e^{-\sqrt{\frac{\gamma}{2\tau}} x} \]

\[ \text{Re } N(0) = \sigma \implies N_0 = \sigma. \]

So finally,

\[ u(t, x) = \text{Re} \left[ e^{i \omega t} - (1+i) \sqrt{\frac{\gamma}{2\tau}} x \right] e^{-\sqrt{\frac{\gamma}{2\tau}} x} \cos \left( \sqrt{\frac{\gamma}{2\tau}} x - \omega t \right) \]

\[ \sqrt{\frac{\gamma}{2\tau}} \omega t = \sqrt{\frac{\gamma}{2\tau}} (x - \sqrt{2\gamma \omega} t) \]

\[ \sqrt{\frac{2\tau}{\gamma}} \] is a "skin depth". Shallower for \( \omega \) legs.