Lecture 8: Fourier series

**Goal:** Represent $f(x)$ as a convergent series:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

**Inner product:**

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx$$

$$\| f \|^2 = \sqrt{\langle f, f \rangle}$$

**Orthogonality:**

$$\langle \cos kx, \cos l x \rangle = \langle \sin kx, \sin l x \rangle = 0, \quad k \neq l$$

$$\langle \cos kx \sin l x \rangle = 0$$

$$\| \cos kx \| = \| \sin kx \| = 1$$

Given this, and not worrying about convergence, we have (if series converge to $f(x)$)

$$a_k = \langle f, \cos kx \rangle, \quad k \geq 0 \quad b_k = \langle f, \sin kx \rangle, \quad k > 0$$
Example: \( f(x) = x \), \(-\pi < x < \pi\)

\[ x \sim 2 \left( \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \ldots \right) \]

Partial sums are called trigonometric polynomials:

\[ s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \]

Convergence means: \( \lim_{n \to \infty} s_n(x) = \tilde{f}(x) \) \( \Leftrightarrow \) not necessarily \( f(x) \).

Let \( f(x) \), \(-\pi < x < \pi\). \( 2\pi \)-periodic extension of \( f \) is unique function \( \tilde{f}(x) \) that satisfies

\[ \tilde{f}(x + 2\pi m) = f(x) \text{, for some } m(x) \in \mathbb{Z}. \]

Example: \( 2\pi \)-periodic extension of \( x \) is sawtooth.
It will turn out to be a better choice to set the function equal to the average at discontinuities:

\[
\tilde{f}(\pi) = \tilde{f}(-\pi) = \frac{1}{2} \left( f(\pi) + f(-\pi) \right)
\]

With this choice, Fourier series converge everywhere to \( f(x) \):

\[
2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin \left( \frac{k\pi}{L} \right) = \begin{cases} x, & -\pi < x < \pi \\ 0, & x = \pm \pi \end{cases}
\]

(Left-hand side is 2π-periodic, by construction.)

Discontinuities in \( f(x) \) and its derivatives are the central aspect that affects convergence. We thus define:

\( f(x) \) is piecewise continuous on \([a, b]\) if it is continuous except possibly at finitely many points

\[a < x_1 < x_2 < \cdots < x_n < b.\]
Furthermore, require that left/right limits exist:

\[ f(x_k^-) = \lim_{x \to x_k^-} f(x), \quad f(x_k^+) = \lim_{x \to x_k^+} f(x) \]

for \( k = 1, \ldots, n \). (One-sided limit at endpoints)

Note that nothing is required of \( f(x_k) \) (could be undefined).

Magnitude of jump \( \beta_k = f(x_k^+) - f(x_k^-) \)

(\( x_k \) is removable if \( \beta_k = 0 \).)

Simpler example: step function \( \sigma(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases} \)

\( \sigma(0^+) - \sigma(0^-) = 1. \)
Now a stronger version:

\( f(x) \) is piecewise \( C^1 \) on \([a, b]\) if it is continuously differentiable except at a finite # of points.

As for piecewise \( C^0 \) with in addition \( f'(x^+) \) existing.

Example: \( f(x) = |x| \) is piecewise \( C^1 \).

Can generalize to piecewise \( C^n \) functions.