Lecture 3: Transport equation (example)

\[ \frac{\partial u}{\partial t} + c(x) \frac{\partial u}{\partial x} = 0 \]
\[ \frac{dx}{dt} = c(x) \]

\[ \beta(x) = \int \frac{dx}{c(x)} = t + k \]

\[ u(t, x) = \int_0^\beta \beta^{-1} (\beta(x) - t) \]

Example:

\[ \frac{\partial u}{\partial t} + \frac{1}{x^2 + 1} \frac{\partial u}{\partial x} = 0 \]

\[ \frac{dx}{dt} = \frac{1}{x^2 + 1} \Rightarrow \beta(x) = \int (x^2 + 1) dx \]
\[ = \frac{1}{3} x^3 + k = t + k \]

Parts of the wave speed up they slow down after they pass \( x = 0 \).
example: \( u_t + (x^2 - 1) u_x = 0 \)

Characteristics: \( \frac{dx}{dt} = x^2 - 1 = c(x) \)

\[
\beta(x) = \int \frac{dx}{x^2 - 1} = \frac{1}{2} \log \left| \frac{x-1}{x+1} \right| = t + \chi
\]

\( x(t) = \frac{1 + e^{2t}}{1 - e^{2t}} \) \( t > 0 \)

Choose: \( x(t) \to 0 \) as \( t \to 0 \)

\( c(x_+) = 0 \)

\( c(x_-) = 0 \)

NOT prescribed by initial data!
\[ x < -1: \quad (\text{actually} \quad |x| > 1) \]

\[ \beta(x) = \frac{1}{2} \log \left( \frac{x-1}{x+1} \right) = t + h \]

\[ \frac{1-x}{1+x} = e^{2(t+h)} \]

\[ \beta^{-1}(t+h) = \chi(t) = \frac{1}{2} \frac{e^{2(t+h)} + 1}{e^{2(t+h)} - 1}, \quad x < -1 \]

The char. for \( \text{data} \to \infty \) as \( t \to 0^+ \) has \( h = 0 \).

With \( u(0, x) = e^{-x^2} = f(x) \)

\[ u(t, x) = \int \beta^{-1}(\beta(x) - t) \]

\[ = \int \frac{x-1 + (x+1)e^{2t}}{1-x + (x+1)e^{2t}} \]

\[ = \int \frac{x+1 + (x-1)e^{-2t}}{x+1 - (x-1)e^{-2t}} \]

\[ \text{crunch against} \sim \text{goes to } \infty \]

\[ \text{not prescribed by initial data} \]
Converges non-uniformly to a step function:

\[ u(t,x) \rightarrow s(x) = \begin{cases} f(1), & x \geq -1 \\ 0, & x < -1 \end{cases} \]

For \( c(x) \) continuously differentiable:

- Unique char. through each \((t,x) \in \mathbb{R}^2\)

- Cannot cross

- \( t = \beta(x) \) char. \( \Rightarrow \) \( t = \beta(x) + h \) also a char.

- Each non-horizontal char. is graph of strictly monotone function. Never reverses direction

- As \( t \) increases, either \( x(t) \stackrel{(t \rightarrow \infty)}{\rightarrow} x \ast \) with \( c(x \ast) = 0 \) or \( x(t) \rightarrow \pm \infty \).