Lecture 24: Boundary layer theory (cont'd)

A quick note about convergence.

Consider
\[ I(x) = x e^x \int_0^\infty \frac{e^{-t}}{t^x} \, dt, \quad x > 0 \]

Let's try to approximate \( I(x) \), for large \( x \).

Repeated integration by parts gives
\[
I(x) = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} + \frac{(-1)^N}{N!} x e^x \int_0^\infty \frac{e^{-t}}{t^{N+1}} \, dt
\]
\[ = S_N(x) + \mathcal{E}(x,N) \]

Now note that ratio test gives \( \left| \frac{(n+1)^{th \ term}}{n^{th \ term}} \right| = \frac{n}{n+1} \)
for \( S_N(x) \), so for fixed \( x \) diverges for all \( x \)!

So throw out \( S_N(x) \)? No!

Observe that \( \mathcal{E}(x,N) > 0 \), \( N \) even,
\[ \leq 0, \quad N \text{ odd} \]
\[ I(x) = S_N(x) + |E(x, N)|, \quad N \text{ even} \]

\[ = S_N(x) - |E(x, N+1)| \]

So

\[ S_N(x) \leq I(x) \leq S_{N+1}(x), \quad N \text{ even} \]

Hold on here: this says that I can approximate \( S(x) \) by partial sums of \( S_N \), even though \( S_N \) diverges as \( N \to \infty \)!

The optimal approximation is the one that minimizes \( E(x, N) \). This error gets smaller as \( x \) gets larger. This gives meaning to

\[ I(x) \approx 1 - \frac{1}{x^c} \]

It means the approximation holds for \( x \) small enough, but adding more terms doesn't necessarily improve the approximation (unless \( x \) is made smaller).

Thus, typically divergent series are useful for approximations.

**Example:** \[ I(100) \approx 0.99019 \]

\[ 1 - \frac{1}{100} = 0.99 \]
Back to boundary layers

Let's familiarize the concepts better.

Consider: $\varepsilon y'' + (1+\varepsilon)y' + y = 0$, \( y(0) = 0 \), \( y(1) = 1 \)

Exact solution: \( y = \frac{e^{-x} - e^{-1/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}} \)

There is a boundary layer of width $\varepsilon$.

Outer limit: \( y_{\text{outer}}(x) = \lim_{\varepsilon \to 0^+} y(x) = e^{1-x} \)

This works at fixed $x$.

Directly in the equation: \( y_{\text{outer}}' + y_{\text{outer}} = 0 \), \( y_{\text{outer}}(1) = 1 \)

For the inner solution take $\varepsilon \to 0$, but for $x$ value always inside the boundary layer.

\( y_{\text{inner}}(x) = Y_{\text{inner}}(X) = \lim_{\varepsilon \to 0^+} y(\varepsilon X) = e^{-e^{1-X}} \)

where $x = \varepsilon X$

\[ \text{thickens of layer} \]"
Directly from equation, rewrite in besss $Y(X) = y(\epsilon x)$:

\[
\frac{1}{\epsilon^2} \frac{d^2 Y}{dX^2} + \left( 1 + \frac{1}{\epsilon} \right) \frac{dY}{dX} + Y = 0
\]

Take $\epsilon \to 0^+$, with $X$ fixed:

\[
\frac{d^2 Y_{in}}{dX^2} + \frac{dY_{in}}{dX} = 0, \quad Y_{in}(0) = 0
\]

$Y_{in} = e^{-x}$ satisfies this.

Note that $\lim_{x \to 0} y_{out}(x) = \lim_{X \to \infty} Y_{in}(X) = e$

In general, the limit is not a number, but some function.

Go to higher order:

\[
y_{out}(x) \sim \sum_{n=0}^{\infty} y_n(x) \epsilon^n, \quad \epsilon \to 0^+\]

\[\uparrow\]

formal asymptotic series

$y_0(1) = 1, \quad y_n(1) = 0, \quad n > 0$.

First find $y_{out}(x)$ perturbatively.
\[ y_0' + y_0 = 0 \quad , \quad y_0(1) = 1 \]
\[ y_n' + y_n = -y_{n-1}' - y_{n-1} \quad , \quad y_n(1) = 0 \quad , \quad n \geq 0 \]

Solution is: \( y_0 = e^{1-x} \), \( y_n = 0 \), \( n \geq 0 \).

So in this case the leading-order approximation from before is correct to all orders in \( \varepsilon \).

\( |y_{n+1} - y_n| \sim O(\varepsilon^n) \) for all \( n \).

Now for the inner solution:

\[ Y_{in}(X) \sim \sum_{n=0}^{\infty} \varepsilon^n Y_n(X), \quad \varepsilon = 0^+ \]

with \( Y_n(0) = 0 \), \( \forall n \).

\[ Y_0'' + Y_0' = 0 \]
\[ Y_n'' + Y_n' = -Y_{n-1}' - Y_{n-1} \]

\[ Y_0(X) = A_0 (1 - e^{-X}) \]
\[ Y_n(X) = \int_0^X (A_n e^{-z} - Y_{n-1}(z)) \, dz, \quad n \geq 0 \]

The \( A_n \) are undetermined constants.
Matching: substitute $x = \varepsilon X$ into $y_{\text{out}}$:

$$y_{\text{out}}(\varepsilon X) = e^{1-\varepsilon X} = e \left[ 1 - \varepsilon X + \frac{\varepsilon^2 X^2}{2!} - \frac{\varepsilon^3 X^3}{3!} + \ldots \right]$$

$$Y_0(X) = A_0 \text{ as } X \to \infty$$

So $A_0 = e$, to match with $y_{\text{out}}(x)$.

$$Y_1(X) = (A_1 + A_0)(1-e^{-X}) - eX$$

$$= A_1 + A_0 - eX \quad \text{as } X \to \infty$$

$$= -eX \quad \text{from matching with } y_{\text{out}} \text{ to order } \varepsilon,$$

So $A_1 = -A_0 = -e$.

etc... Get eventually

$$Y_n(X) = e \sum_{n=0}^{\infty} \varepsilon^n (-1)^n \frac{X^n}{n!} - e^{1-X}$$

Uniformly valid solution:

$$= e^{1-X} - e^{-e^{1-X/\varepsilon}}$$

$$y_{\text{unif}} = y_{\text{in}} + y_{\text{out}} - y_{\text{match}}$$

$$= (e^{1-X} - e^{-X/\varepsilon}) + (e^{1-X}) - (e^{1-X})$$

$$= e \left[ e^{-X} - e^{-X/\varepsilon} \right]$$

Not the same as exact solution!

To all orders in $\varepsilon$
\[ \frac{y_{\text{exact}}}{y_{\text{uni}}} = \left( \frac{e^{-x} - e^{-x/\epsilon}}{e^{-1} - e^{-1/\epsilon}} \right) \frac{e^{-1}}{e^{-x} - e^{-x/\epsilon}} \]

\[ = \left( 1 - e^{1-\epsilon} \right)^{-1} \]

\[ \sim 1 + e^{1-\epsilon} \]

Correction is "beyond all orders"