Lecture 23: Boundary-layer theory

Example: $(x-3y)y' + xy = e^{-x}, \quad y(1) = 1/2$, find $y(0)$

Set $\varepsilon = 0$: $x(y' + y) = e^{-x}$

$$y = (1 + \log x) e^{-x}$$

But as $x \to \infty$, $y \to \infty$, so $\varepsilon y' \to \infty$.

There is a boundary layer near $x = 0$.

Split in two regions: "inner" (inside boundary layer)

"outer" (the rest)

$$y_{out} = (1 + \log x) e^{-x} \quad \text{since} \quad y(y = 1/2 \text{ is in outer region).}$$

Assume boundary layer of width $\delta$.

$\varepsilon y_{out} \approx \varepsilon \ln \delta \left( \frac{1}{\delta} \right) \sim \mathcal{O}(1), \quad \therefore \frac{\delta}{\log \delta} \sim \mathcal{O}(\varepsilon)$

Note that $\frac{\delta}{\log \delta} \to 0$ as $\varepsilon \to 0$

In the inner region $x \ll \delta$, we shall have:
\[(x - \varepsilon y_{in}) y_{in}' = 1\]

We used: \[\frac{x y}{y'} \sim \frac{y}{y'} \ll 1\] since \(y\) blows as \(\varepsilon \to 0\).

Solution: \[x = \varepsilon (y_{in} + 1) + Ce^{y_{in}}\]

Can use this to determine \(y(0)\).

To find \(C\), use asymptotic matching.

Take \(x\) small, but not as small as \(\varepsilon\): \(x = O(\varepsilon^{1/2})\)

\[y_{in} \sim 1 + \ln x\]
\[x \sim Ce^{y_{in}}\]

Thus: \(y_{in}(0)\) satisfies \[0 = \varepsilon(y_{in}(0) + 1) + \varepsilon\]

Clearly, \(y_{in}(0) \to -\infty\) as \(\varepsilon \to 0\).

Let \(\alpha = -y_{in}(0): \quad 0 = -\varepsilon(\alpha - 1) + \varepsilon^{-\alpha - 1}\)

To leading order: \[\varepsilon \alpha = e^{-\alpha}\]
\[\log \varepsilon + \log \alpha = -\alpha\]

So \(\alpha \sim -\log \varepsilon\) as \(\alpha \to 0\).
\[ \xi = -\log \varepsilon + b. \]

\[ \varepsilon (-\log \varepsilon + b - 1) = e \]

\[ \log \varepsilon + \log (-\log \varepsilon + b - 1) = \log \varepsilon - (b + 1) \]

\[ \log \left[ \frac{\log \varepsilon}{\log \varepsilon} \left( 1 + \frac{b - 1}{\log \varepsilon} \right) \right] = -(b + 1) \]

\[ \log \left| \frac{\log \varepsilon}{\log \varepsilon} + \frac{b - 1}{\log \varepsilon} + O\left( \frac{b}{\log \varepsilon} \right) \right| = -(b + 1) \]

So \( b + 1 = -\log \left| \log \varepsilon \right| \)

Check: \( \frac{\log \left| \log \varepsilon \right|}{\log \varepsilon} \ll \frac{\log \left| \log \varepsilon \right|}{\log \varepsilon} \) self-consistent

Conclude: \( y_1(b) = \log \varepsilon + \log \left| \log \varepsilon \right| + 1 + \ldots \)

For \( \varepsilon = 0.01 \), this is \( \approx -2.078 \).

The numerical solution gives \(-2.942\). slow convergence

This might seem like a big error, but the only other guess we had for \( y(0) \) was \(-\infty\)!
Example: \( \varepsilon y'' + a(x)y' + b(x)y = 0 \quad 0 \leq x \leq 1 \)
\( y(0) = A, \ y(1) = B \)

Assume \( a(x) \neq 0, \ 0 \leq x \leq 1 \) (as lab), \( a(x) > 0 \)
Otherwise \( a, b \) continuous.

Boundary layer as \( x \to 0 \).

Outer: \( a y'' + b y = 0 \)

\( y_{in} = B \exp \left[ \int_0^1 \frac{b(t)}{a(t)} \, dt \right] \)

Valid as \( \varepsilon \to 0 \) for \( 5 \ll x \leq 1 \).

If \( y_{in}(0) = A \), we are done, but this is exceptional.

Near \( x = 0 \), approximate \( a(x) \approx \alpha, \ b(x) \approx \beta \).

Also, \( y/y \to \infty \) since \( y \) varies rapidly.

Hence, \( \varepsilon y'' + \alpha y' = 0 \)

\( y_{in} = A + C (e^{-\alpha x/\varepsilon} - 1) \)

Now for the asymptotic matching.
The boundary layer is of thickness $s \sim \varepsilon$

\[
\frac{\varepsilon y''}{\alpha y'} \sim \varepsilon \delta \sim O(1) \Rightarrow s \sim \varepsilon
\]

Matching region: take $x \sim \varepsilon^{1/2}$, say $(\varepsilon \ll x \ll 1)$

\[
y_{in}(x) \sim A - C \quad \text{matching depends on } \alpha > 0
\]

\[
y_{out}(x) \sim B \exp \left[ \int_{0}^{1} \frac{b(t)}{c(t)} \, dt \right] = y_{out}(0)
\]

Hence, $C = A - y_{out}(0)$, and

\[
y(x) \sim B \exp \left[ \int_{0}^{1} \frac{b(t)}{c(t)} \, dt \right], \quad \varepsilon \ll x \ll 1
\]

\[
\sim A e^{-\frac{a(0)x}{\varepsilon}} + B \left(1-e^{-\frac{a(0)x}{\varepsilon}}\right) \exp \left[ \int_{0}^{1} \frac{b(t)}{c(t)} \, dt \right]
\]

$x = 0(\varepsilon)$

Can combine using,

\[
y_{unif} = y_{out} + y_{in} - y_{match}
\]

\[
y_{unif} = B \exp \left[ \int_{0}^{1} \frac{b(t)}{c(t)} \, dt \right] + \left\{ A - B \exp \left[ \int_{0}^{1} \frac{b(t)}{c(t)} \, dt \right] \right\} e^{-\frac{a(0)x}{\varepsilon}}
\]
This is a “uniform” approximation in the sense that:

$$\left| y(x) - y_{\text{unit}}(x) \right| \sim o(\varepsilon) \quad 0 \leq x \leq 1, \quad \varepsilon \to 0^+$$

Observe: if $a(x) < 0$, the boundary layer is at $y = 1$. 