Lecture 22: Asymptotic matching

Asymptotic matching can also be used to expand integrals.

example: $$I(x) = \int_0^{\pi/2} e^{ix\cos t}\,dt \quad , \quad x \gg 1$$

Last time we used method of stationary phase to obtain leading-order behavior in x. Only depends on integrand values near stationary point $t=0$. Upper bound doesn't matter.

To get next order, need global analysis.

Introduce $0 < \delta(x) \ll 1$:

$$I(x) = \underbrace{\int_0^{\delta} e^{ix\cos t}\,dt}_{I_1(x)} + \underbrace{\int_{\delta}^{\pi/2} e^{ix\cos t}\,dt}_{I_2(x)} \quad , \quad x \gg 1$$

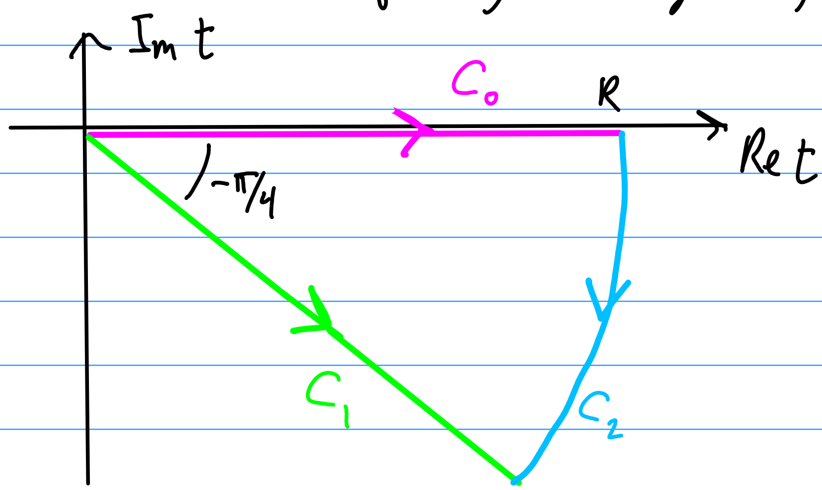
For I_1, since $\delta \ll 1$, $\cos t = 1 - \frac{t^2}{2} + O(\delta^4)$

$$I_1(x) = e^{ix} \underbrace{\int_0^{\delta} e^{-ixt^2/2}\,dt}_{\sim \delta} + \underbrace{O(x\delta^5)}_{\sim \int_0^{\delta} x t^4\,dt}$$

So need $x\delta^4 \to 0$ as $x \to \infty$.

$$\int_0^\delta e^{-ixt^2/2}\,dt = \underbrace{\int_0^\infty e^{-ixt^2/2}\,dt}_{1} - \underbrace{\int_\delta^\infty e^{-ixt^2/2}\,dt}_{2}$$

Rotate contour of integration by $-\pi/4$, so $t^2 \sim -is^2$



$$C = C_0 - C_1 + C_2$$

$$\int_C e^{ix\cos t}\,dt = 0$$

$$1 = \lim_{R\to\infty} \int_{C_0} e^{-\frac{1}{2}ixt^2}\,dt$$

Show: $\displaystyle\lim_{R\to\infty} \int_{C_2} e^{-\frac{1}{2}ixt^2}\,dt = 0.$ Let $t = Re^{i\theta}$

$\theta: 0 \to -\pi/4$

$$\lim_{R\to\infty}\left|\int_0^{-\pi/4} e^{-\frac{1}{2}ixR^2 e^{2i\theta}} R i e^{i\theta}\,d\theta\right| \le \lim_{R\to\infty} \int_{-\pi/4}^{0} R e^{\frac{1}{2}xR^2 \underbrace{\sin 2\theta}_{\le 0}}\,d\theta$$

$$= 0$$

Hence,
$$\underbrace{\int_0^\infty e^{-ixt^2/2}\,dt}_{C_0} = e^{-i\pi/4} \underbrace{\int_0^\infty e^{-xs^2/2}\,ds}_{C_1} = \sqrt{\frac{\pi}{2x}}\, e^{-i\pi/4}$$

Now for 2 $= \displaystyle\int_\delta^\infty e^{-ixt^2/2}\,dt$

The fact that 1 converges establishes convergence for 2.

Use integration by parts:

$$\int_a^b f(t)\, e^{-\phi(t)}\,dt = \int_a^b \left(\frac{-f(t)}{\phi'(t)}\right) d\left(e^{-\phi(t)}\right)$$

(Requires $\phi' \neq 0$, so can't use in 1, hence stationary point)

$$= \left.\frac{-e^{-\phi(t)}}{\phi'(t)} f(t)\right|_a^b + \int_a^b e^{-\phi(t)}\, d\left(\frac{f(t)}{\phi'(t)}\right)$$

$$= \left.\frac{-e^{-\phi(t)}}{\phi'(t)} f(t)\right|_a^b + \int_a^b \left(\frac{f'(t)\phi'(t) - f(t)\phi''(t)}{(\phi'(t))^2}\right) e^{-\phi(t)}\,dt$$

as long as $\phi'(t) \neq 0$ for $a \leq t \leq b$

no stationary point

$$\int_\delta^\infty e^{-ixt^2/2}\,dt = \frac{e^{-ixt^2/2}}{ix\delta} + \frac{i}{x}\int_\delta^\infty \frac{e^{-ixt^2/2}}{t^2}\,dt$$

$\phi(t) = ixt^2/2$

$\phi'(t) = ixt$

Then let $f = \frac{1}{t^2}$, $f' = \frac{-2}{t^3}$

$$2 = \frac{e^{-ix\delta^2/2}}{ix\delta} + \frac{i}{x}\left[\left(\frac{1}{\delta^2}\right)\left(\frac{1}{ix\delta}\right)e^{-ix\delta^2/2} + \int_\delta^\infty \left(\frac{(-2/t^3)(ixt) - \frac{1}{t^2}(ix)}{-x^2t^2}\right) \times e^{-ixt^2/2}\,dt\right]$$

$$2 = -i\frac{e^{-ix\delta^2/2}}{x\delta} + \frac{e^{-ix\delta^2/2}}{x^2\delta^3} - \frac{3}{x^2}\int_\delta^\infty \frac{e^{-ixt^2/2}}{t^4}\,dt$$

Now put $f = \frac{1}{t^4}$, $f' = \frac{-4}{t^3}$:

$$2 = -i\frac{e^{-ix\delta^2/2}}{x\delta} + \frac{e^{-ix\delta^2/2}}{x^2\delta^3} - \frac{3}{x^2}\left[\frac{e^{-ix\delta^2/2}}{ix\delta^5} + \frac{5i}{x}\int_\delta^\infty \frac{e^{-ixt^2/2}}{t^5}\,dt\right]$$

$$= -i\frac{e^{-ix\delta^2/2}}{x\delta} + \frac{e^{-ix\delta^2/2}}{x^2\delta^3} + \frac{3i\,e^{-ix\delta^2/2}}{x^3\delta^5} + O\left(\frac{1}{x^4\delta^7}\right)$$

Conclude finally:

$$I_1(x) = \sqrt{\frac{\pi}{2x}}\,e^{i(x-\pi/4)} + \left(\frac{i}{x\delta} - \frac{1}{x^2\delta^3} - \frac{3i}{x^3\delta^5}\right)e^{ix(1-\delta^2/2)}$$

$$+ O(x\delta^5) + O\left(\frac{1}{x^4\delta^7}\right)$$

The ratio of terms (in integration by parts) $\sim \frac{1}{x\delta^2} \to 0$, so require $\boxed{x\delta^2 \to \infty}$.

Also, need

$$\frac{x\delta^5}{\left(\frac{1}{x^3\delta^5}\right)} \sim x^4\delta^{10} \ll 1, \text{ so require } \boxed{x^2\delta^5 \ll 1}$$

more stringent than $x\delta^4 \ll 1$

$(x^{2/5}\delta \ll 1$ vs $x^{1/4}\delta \ll 1)$

Both conditions can be satisfied for

$$x^{-1/2} \ll \delta \ll x^{-2/5}, \quad x \to \infty$$

Now for $I_2(x) = \int_\delta^{\pi/2} e^{ix\cos t}\, dt$

No stationary points in integrand, so integration by parts works fine:

$$I_2(x) = \frac{i}{x} - \frac{ie^{ix\cos\delta}}{x\sin\delta} + \frac{e^{ix\cos\delta}\cos\delta}{x^2\sin^3\delta}$$

$$+ \frac{ie^{ix\cos\delta}(2\cos^2\delta+1)}{x^3\sin^5\delta} + O\left(\frac{1}{x^3}\right) + O\left(\frac{1}{x^4\delta^7}\right)$$

Require $x^{2/5}\delta \to 0$ as before, as well as

$$\frac{\frac{1}{x^4\delta^7}}{\frac{1}{x}} \sim \frac{1}{x^3\delta^7} \ll 1, \quad \text{so } \boxed{x^{3/7}\delta \to \infty}$$

(more stringent than $x^{1/2}\delta \to \infty$)

Now add I_1 and I_2:

$$I_1 + I_2 = \sqrt{\frac{\pi}{2x}}\, e^{i(x-\pi/4)} + \frac{i}{x} + O\left(\frac{1}{x^4\delta^7}\right) + O(x\delta^5) + O\left(\frac{1}{x^3}\right)$$

$$x \to \infty, \quad x^{2/5}\delta \to 0, \quad x^{3/7}\delta \to \infty$$

We have our first two terms!

No δ.

Check error: (for consistency) $\dfrac{\frac{1}{x^4\delta^7}}{\frac{1}{x}} \sim \dfrac{1}{x^3\delta^7} \to 0$ since $x^{3/7}\delta \to \infty$.

$$\frac{x\delta^5}{\frac{1}{x}} \sim x^2\delta^5 \to 0 \text{ since } x^{2/5}\delta \to 0$$

$$\frac{1/x^3}{1/x} \sim \frac{1}{x^2} \to 0 \text{ since } x \to \infty$$

Require $x^{-3/7} \ll \delta \ll x^{-2/5}$ barely fits!

or $x^{-15/35} \ll \delta \ll x^{-14/35}$ as $x \to \infty$

So why 3 integration by parts? The correction i/x is given after 1 integration.

The consistency condition would have been wrong:

After one integration by parts:

$$I_2(x) = \frac{i}{x} - \frac{ie^{ix\cos\delta}}{x\sin\delta} + O\left(\frac{1}{x^3}\right) + O\left(\frac{1}{x^2\delta^3}\right)$$

So need $\dfrac{\left(\frac{1}{x^2\delta^3}\right)}{\frac{1}{x}} = \dfrac{1}{x\delta^3} \ll 1$, so $\delta \gg x^{-1/3}$.

But also need $\delta \ll \delta^{-2/5}$ as before, so

$$x^{-1/3} \ll \delta \ll x^{-2/5}$$

Impossible!

After two integration by parts:

$$I_2(x) = \frac{i}{x} - \frac{i e^{ix\cos\delta}}{x\sin\delta} + \frac{e^{ix\cos\delta}\cos\delta}{x^2\sin^3\delta} + O\left(\frac{1}{x^3}\right) + O\left(\frac{1}{x^3\delta^5}\right)$$

Need: $$\frac{\left(\frac{1}{x^3\delta^5}\right)}{\left(\frac{1}{x}\right)} = \frac{1}{x^2\delta^5} \ll 1, \text{ so } \delta \gg x^{-2/5}$$

So need $$x^{-2/5} \ll \delta \ll x^{-2/5}$$

Again, impossible!

Hence, need 3 integrations to show δ exists.