Supplement: Legendre's equation

\[ \left( (1-x^2) \phi' \right)' + x \phi = 0 \quad , \quad -1 < x < 1 \]

\[ s(x) = 1 - x^2 \quad , \quad p(x) = q(x) = 1 \]

Eigen solutions with different \( l \)'s are orthogonal, since \( s(\pm 1) = 0 \).

So by series:
\[ \phi(x) = \sum_{n=0}^{\infty} a_n x^n \]
\[ \phi'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \]
\[ \phi''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \]

Substitute in:
\[ (1-x^2) \phi'' - 2x \phi' + \lambda \phi = 0 \]

Need to relabel \( \phi'' \) sum: let \( n' = n-2 \)
\[ \phi''(x) = \sum_{n'=-2}^{\infty} (n'+1)(n'+2) a_{n'+2} x^{n'} \]

Now drop the prime on \( n' \), and start sum at 0 since first two terms vanish anyways.
Combining the given, we have
\[(1-x^2)\phi'' - 2x\phi' + \lambda \phi\]
\[= \sum_{n=0}^{\infty} \left[(n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + \lambda a_n\right] x^n = 0\]

Since \(\phi'(0) = 0\) (assuming convergent), set coefficient of \(x^n\) to 0:

\[(n+1)(n+2) a_{n+2} - n(n-1) a_n - 2n a_n + \lambda a_n = 0\]

\[a_{n+2} = \frac{n(n-1) + 2n - \lambda}{(n+1)(n+2)} a_n\]

Recurrence relation for coefficients

Two solutions: \(\phi_0, \phi_2, \phi_4, \ldots\) (even)
\[\phi_1, \phi_3, \phi_5, \ldots\] (odd)

But do these converge? Ratio test.

\[R_n = \left| \frac{a_{n+2} x^{n+2}}{a_n x^n} \right| = \left| \frac{n(n+1) - \lambda}{(n+1)(n+2)} \right| |x|^2\]
\[ \lim_{n \to \infty} R_n = R = |x|^2. \]

So converge absolutely for \( |x| < 1 \).

What about \( |x| = 1 \)? More difficult.

For \( n \) large, note that

\[ a_{n+2} = \frac{n(n+1)-1}{(n+1)(n+2)} a_n \approx \frac{n}{n+2} a_n \]

So \( a_{n+4} \approx \frac{n+2}{n+4} a_{n+2} = \frac{n+2}{n+4} \frac{n}{n+2} a_n = \frac{n}{n+4} a_n \)

\[ a_{n+6} \approx \frac{n+4}{n+6} \frac{n}{n+4} a_n = \frac{n}{n+6} a_n \]

Easy to see: \( a_{n+2m} \approx \frac{n}{n+2m} a_n \)

So as \( m \to \infty \), \( a_{n+2m} \sim \frac{1}{2m} \text{ divergent} \)

\[ \sum a_n x^n \] is thus a divergent series at \( |x| = 1 \),

since it behaves like harmonic series \( \sum \frac{1}{n} \).

(Signs don't alternate at \( x = -1 \), since only even/odd powers.)
We conclude for each \( \lambda \) we have two independent solutions (even, odd), but these diverge at \( |x| = 1 \) (one or both).

(This is tied to the fact that \((1-x^2)\phi''+...\) has a vanishing coefficient for \(x(=1)\).)

Hence, we cannot construct regular solutions for general \( \lambda \).

**But:** if \( \lambda = m(m+1), \ m = 0, 1, 2, 3, \ldots \)

then the series terminates when \( n = m! \).

\( \Rightarrow \) Legendre polynomials.

When \( m \) is even, \( a_0 + a_2 x^2 + \ldots + a_m x^m \) terminates.

When \( m \) is odd, \( a_1 x + a_3 x^3 + \ldots + a_m x^m \) terminates.

For instance, for \( m = 0 \), we have \( \phi_{0m}(x) = 1 \) as a solution. The other solution is given by the odd series:

\[
\begin{align*}
\alpha_{n+2} &= \frac{n}{n+2} a_n, \quad \alpha_1 = 1 \\
\alpha_{1+2m} &= \frac{1}{1+2m} a_1 \quad \Rightarrow \quad \alpha_{2m+1} = \frac{1}{2m+1}
\end{align*}
\]

\( m = 0: \quad \phi_{0m}(x) = 1 \), \( \phi_{odd}(x) = \sum_{m=0}^{\infty} \frac{x^{2m+1}}{2m+1} \)
So, for \( m = 0, 1, 2, 3, \ldots \), two solutions:

\( P_m(x) \) are Legendre polynomials (L. functions of first kind)

\( Q_m(x) \) are Legendre functions of the second kind

In practical problems, we usually throw out \( Q_m(x) \) since we found regular (barrier) solution in \([-1, 1]\).

Why do some solutions of Legendre's equation diverge?

\[(1-x^2)\phi'' - 2x\phi' + \lambda\phi = 0\]

This is 0 at \( x = \pm 1 \). Allows \( \phi'' \to 0 \), with \((1-x^2)\phi''\) finite.

Let's examine the blowup at \( x = 1 \). Let \( y = 1-x \) or \( x = 1-y \).

Then

\[(2-y)y\phi'' + 2(1-y)\phi' + \lambda\phi = 0\]

If \( \phi \) blows up as \( y \to 0^+ \), so do \( \phi', \phi'' \).

In fact \( \phi' \) blows up faster than \( \phi \).

Let's show this. First prove:

Lemma: If \( f(x) \) is continuously differentiable in \([a, b]\), \( b > a \), and \( \lim_{x \to b^-} f(x) = \infty \), then \( \lim_{x \to b^-} f'(x) = \infty \).
proof: Assume $f'(x) < M$, $x \in [a, b]$.

The mean value theorem says $f'(c) = \frac{f(x) - f(a)}{x - a}$, $a \leq x < b$, for some $c \in [a, b]$. But $f'(c) < M$, so

$$\frac{f(x) - f(a)}{x - a} < M \Rightarrow f(x) < f(a) + M(x - a).$$

But this implies $f(x)$ is bounded as $x \to b^-$, which contradicts the assumption. \[ \square \]

Corollary: $\lim_{x \to b^-} f' = \infty$

proof: $\lim_{x \to b^-} f' = \lim_{x \to b^-} (\log f)' = \lim_{x \to b^-} y'$, $y = \log f$.

But $y$ has $\lim_{x \to b^-} y = \infty$, so $\phi$ itself goes to $\infty$.

Then by the Lemma $\lim_{x \to b^-} y' = \infty$.

The Corollary says that $\phi'$ goes to $\infty$ infinitely faster than $\phi$.

In the same way, $\phi'' \to \infty$ faster than $\phi'$.

Thus: $|\phi''| > |\phi'| > |\phi|$. 
Now back to Legendre's equation in the coordinate $y = 1 - x$:

$$(2-y)y\phi'' + 2(1-y)\phi' + 2\phi = 0$$

If $\phi \rightarrow \infty$ as $y \rightarrow 0^+$, then so do $\phi$, $\phi'$, $\phi''$.

The largest terms in the equation, as $y \rightarrow 0^+$, are

$$2y\phi'' + 2\phi' = 0, \quad y \rightarrow 0^+$$

(At fixed $\lambda$, there is no way $2\phi$ is as large as $\phi'$.)

Hence:

$$y\phi'' + \phi' = 0 \implies \phi' \sim \frac{1}{y}$$

or

$$\phi(y) \sim \log y$$

Hence, the series solution diverges logarithmically as $y \rightarrow 0^+$ ($x \rightarrow \pm 1$).

This is consistent with the singularity being linked to the harmonic series, which diverge logarithmically:

$$\log(N+1) < \sum_{n=1}^{N} \frac{1}{n} \leq 1 + \log N$$

$$\frac{1}{n} \leq \int_{n}^{n+1} \frac{dx}{x} < \sum_{n=1}^{N} \frac{1}{n}$$

(Comes from this...