Math 322: Midterm Exam

30 October 2018

Name: solutions

Instructions:

- This exam consists of 8 pages, including this one. If pages are missing, let the examiner know right away.
- **Do not unstaple the pages.**
- Attempt all THREE (3) questions.
- **Show all relevant work.**
- Answer each question on its own page, turning the page over if you need more space.
- If you need more paper, raise your hand and more will be provided.

Score: 1 ___ 2 ___ 3 ___ TOTAL ____ / 100

QUESTION 1 (____ / 30)

Find the separated solutions $u(x,t)$ of the heat equation $u_t = ku_{xx}$ in the region $0 < x < L, t > 0$ that satisfy the boundary conditions $u(0,t) = 0$, $u_x(L,t) = 0$. (Justify the possible signs of the eigenvalues.)

ANSWER

$u_t = ku_{xx}$ $\quad u(x,t) = \phi(x)h(t)$ separated form

$\phi h' = k\phi'' h$ $\quad h' + \lambda k h = 0 \rightarrow h(t) = e^{-\lambda k t}$

$\frac{h'}{kh} = \frac{\phi''}{\phi} = -\lambda \Rightarrow \phi'' + \lambda\phi = 0, \ \phi(0) = 0, \ \phi'(L) = 0$

$\underline{\lambda = 0}$: $\phi'' = 0 \Rightarrow \phi = A + Bx$. $\quad \phi(0) = 0 \Rightarrow A = 0$, $\phi'(L) = B = 0 \Rightarrow B = 0$ $\quad$ <u>trivial</u>

$\underline{\lambda < 0}$: Let $\lambda = -\mu^2$. $\phi'' - \mu^2\phi = 0$

$\Rightarrow \phi(x) = A\cosh(\mu x) + B\sinh(\mu x)$

$\phi(0) = A = 0$

$\phi'(L) = B\mu\cosh(\mu L) = 0 \Rightarrow B = 0$ since $\cosh x \neq 0$ for all x.

<u>trivial</u>

$\underline{\lambda > 0}$: Let $\lambda = +\mu^2$: $\phi'' + \mu^2\phi = 0$

$\Rightarrow \phi(x) = A\cos(\mu x) + B\sin(\mu x)$. $\quad \phi(0) = A = 0$

$\phi'(L) = B\mu\cos(\mu L) = 0 \Rightarrow \mu = (n + \frac{1}{2})\frac{\pi}{L}, \ n = 1, 2, 3\ldots$

Conclude: separated solutions are of the form

$$\boxed{u(x,t) \sim e^{-\mu_n^2 k t}\cos(\mu_n x), \quad \mu_n = (n + \tfrac{1}{2})\frac{\pi}{L}, \ n = 1, 2, 3\ldots}$$

(continue on the next page if you need more space)

QUESTION 2 (____ / 35)

(a) Show by direct integration the orthogonality relation

$$\int_{-L}^{L} \cos(m\pi x/L)\cos(n\pi x/L)\,dx = C_m\,\delta_{mn}$$

where $m \geq 0$ and $n \geq 0$ are integers. Find the value of the constant C_m. Justify all your steps. *Hint: You may use the trig identity* $\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b))$.

(b) Find the coefficients of the Fourier series of the function $f(x) = \cos(\pi x/(2L))$, $-L \leq x \leq L$.

(c) What value does the Fourier series in (b) converge to at $x = L$? Justify your answer.

ANSWER

(a) $\underline{m=0:}$ $\displaystyle\int_{-L}^{L} \cos(0)\cos\left(\frac{n\pi x}{L}\right)dx = \left.\frac{\sin\left(\frac{n\pi x}{L}\right)}{(n\pi/L)}\right|_{-L}^{L} = 0, \quad n \neq 0$

$$= \int_{-L}^{L} 1\cdot dx = 2L, \quad n = 0.$$

$$= C_0\,\delta_{0n}, \quad C_0 = 2L.$$

$\underline{m \neq 0:}$ $\displaystyle\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right)\right]dx$

$$= \left.\frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{2(m+n)\pi/L}\right|_{-L}^{L} + \left.\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{2(m-n)\pi/L}\right|_{-L}^{L} = 0, \quad n \neq m$$

$$= 0 + \frac{1}{2}\int_{-L}^{L} 1\cdot dx = L, \quad n = m$$

$$= C_m\,\delta_{mn} \Rightarrow C_m = L$$

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(b) $f(x) = \cos\left(\frac{\pi x}{2L}\right), \quad -L \le x \le L$

$$f(x) \sim \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$B_n \to 0$ (f is even)

$$\int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = C_m A_m + 0$$

(using (c)) (integral of odd·even)

$$A_0 = \frac{1}{C_0} \int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right) dx = \frac{1}{2L} \left. \frac{\sin\left(\pi x / 2L\right)}{(\pi/2L)} \right|_{-L}^{L}$$

$$= \frac{2}{\pi} \sin(\pi/2) = \boxed{\frac{2}{\pi}}$$

$$A_m = \frac{1}{C_m} \int_{-L}^{L} \cos\left(\frac{\pi x}{2L}\right) \cos\left(\frac{m\pi x}{L}\right) dx$$

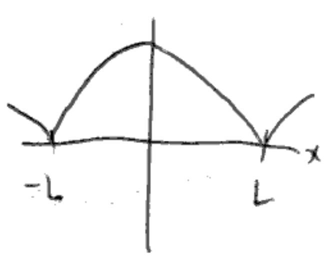
$$= \frac{1}{2L} \int_{-L}^{L} \left[\cos\left(\left(m+\tfrac{1}{2}\right)\frac{\pi x}{L}\right) + \cos\left(\left(m-\tfrac{1}{2}\right)\frac{\pi x}{L}\right) \right] dx$$

$$= \frac{1}{2L} \left[\frac{\sin\left(\left(m+\frac{1}{2}\right)\frac{\pi x}{L}\right)}{\left(m+\frac{1}{2}\right)\pi/L} + \frac{\sin\left(\left(m-\frac{1}{2}\right)\frac{\pi x}{L}\right)}{\left(m-\frac{1}{2}\right)\pi/L} \right]_{-L}^{L}$$

$$= \frac{1}{\pi} \left[\frac{\sin\left(m+\frac{1}{2}\right)\pi}{m+\frac{1}{2}} + \frac{\sin\left(m-\frac{1}{2}\right)\pi}{m-\frac{1}{2}} \right] = \frac{1}{\pi} \left[\frac{(-1)^m}{m+\frac{1}{2}} - \frac{(-1)^m}{m-\frac{1}{2}} \right]$$

$$\boxed{= -\frac{1}{\pi} \frac{(-1)^m}{m^2 - \frac{1}{4}}}$$

(c)



Since $f(L) = f(-L)$, $\tilde{f}(x)$ (periodic extension) is continuous. So series converges to $f(L) = 0$.

$$\psi'' - \lambda_n \psi = 0 \Rightarrow \psi_n = A\cosh(\sqrt{\lambda_n}\, y) + B\sinh(\sqrt{\lambda_n}\, y)$$

$\lambda_n > 0$

$$\psi_n(0) = A = 0.$$

Summing over separated solutions:

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

Find the A's by applying BC at $y=H$:

$$u(x,H) = f(x) = A_0 H + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi H}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

Let $f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$

$$a_0 = \frac{1}{\cancel{2}L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{\cancel{2}L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \underline{n>0}$$

Then $\cancel{A}$ $A_0 = \frac{a_0}{H}$, $A_n = \dfrac{a_n}{\sinh\left(\frac{n\pi H}{L}\right)}$

$$\boxed{u(x,y) = \frac{a_0 y}{H} + \sum_{n=1}^{\infty} a_n \frac{\sinh\left(\frac{n\pi y}{L}\right)}{\sinh\left(\frac{n\pi H}{L}\right)}\cos\left(\frac{n\pi x}{L}\right)}$$