as \( w \to 0 \). As \( w \to \infty \), the behavior of \( I_m(w) \) will be some linear combination of the two different asymptotic behaviors \( (e^{\pm w/w^{1/2}}) \). In general, it will be composed of both and hence is expected to exponentially grow as \( w \to \infty \). In more advanced works, it is shown that

\[
I_m(w) \sim \sqrt{\frac{1}{2\pi w}} e^w, \tag{7.9.44}
\]

as \( w \to \infty \). The most important facts about this function is that \( I_m(w) \) is well behaved at \( w = 0 \) but grows exponentially as \( w \to \infty \).

Some modified Bessel functions are sketched in Fig. 7.9.2. Although we have not proved it, note that both \( I_m(w) \) and \( K_m(w) \) are not zero for \( w > 0 \).

![Figure 7.9.2 Various modified Bessel functions (from Abramowitz and Stegun [1974]).](image)

**EXERCISES 7.9**

7.9.1. Solve Laplace’s equation inside a circular cylinder subject to the boundary conditions

(a) \( u(r, \theta, 0) = \alpha(r, \theta), \quad u(r, \theta, H) = 0, \quad u(a, \theta, z) = 0 \)

(b) \( u(r, \theta, 0) = \alpha(r) \sin \theta, \quad u(r, \theta, H) = 0, \quad u(a, \theta, z) = 0 \)

(c) \( u(r, \theta, 0) = 0, \quad u(r, \theta, H) = \beta(r) \cos 3\theta, \quad \frac{\partial u}{\partial r}(a, \theta, z) = 0 \)

(d) \( \frac{\partial u}{\partial z}(r, \theta, 0) = \alpha(r) \sin 3\theta, \quad \frac{\partial u}{\partial z}(r, \theta, H) = 0, \quad \frac{\partial u}{\partial r}(a, \theta, z) = 0 \)

(e) \( \frac{\partial u}{\partial z}(r, \theta, 0) = \alpha(r, \theta), \quad \frac{\partial u}{\partial z}(r, \theta, H) = 0, \quad \frac{\partial u}{\partial r}(a, \theta, z) = 0 \)

For (e) only, under what condition does a solution exist?
7.9. Laplace's Equation in a Circular Cylinder

7.9.2. Solve Laplace's equation inside a semicircular cylinder, subject to the boundary conditions

(a) \[ u(r, \theta, 0) = 0, \quad u(r, \theta, H) = \alpha(r, \theta), \quad u(r, 0, z) = 0, \]
\[ u(r, \pi, z) = 0, \quad u(a, \theta, z) = 0 \]

(b) \[ u(r, \theta, 0) = 0, \quad \frac{\partial u}{\partial z}(r, \theta, 0) = 0, \quad u(r, 0, z) = 0, \]
\[ u(r, \pi, z) = 0, \quad u(a, \theta, z) = \beta(\theta, z) \]

(c) \[ \frac{\partial}{\partial z} u(r, \theta, 0) = 0, \quad \frac{\partial}{\partial z} u(r, \theta, H) = 0, \quad \frac{\partial u}{\partial \theta}(r, \theta, 0) = 0, \]
\[ \frac{\partial}{\partial \theta} (r, \pi, z) = 0, \quad \frac{\partial u}{\partial r}(a, \theta, z) = \beta(\theta, z) \]

For (c) only, under what condition does a solution exist?

(d) \[ u(r, \theta, 0) = 0, \quad u(r, 0, z) = 0, \quad u(a, \theta, z) = 0, \]
\[ u(r, \pi, H) = \alpha(r, \theta), \quad u(r, 0, z) = 0, \]
\[ u(a, 0, z) = 0, \quad u(r, 0, H) = 0, \quad \frac{\partial u}{\partial \theta}(a, 0, z) = \beta(\theta, z) \]

7.9.3. Solve the heat equation \[ \frac{\partial u}{\partial t} = k\nabla^2 u \]
inside a quarter-circular cylinder \((0 < \theta < \pi/2 \text{ with radius } a \text{ and height } H)\) subject to the initial condition

\[ u(r, \theta, z, 0) = f(r, \theta, z) \]

Briefly explain what temperature distribution you expect to be approached as \(t \to \infty\). Consider the following boundary conditions

(a) \[ u(r, \theta, 0) = 0, \quad u(r, \theta, H) = 0, \quad u(r, 0, z) = 0, \]
\[ u(r, \pi/2, z) = 0, \quad u(a, \theta, z) = 0 \]

(b) \[ \frac{\partial u}{\partial z}(r, \theta, 0) = 0, \quad \frac{\partial u}{\partial z}(r, \theta, H) = 0, \quad \frac{\partial u}{\partial \theta}(r, 0, z) = 0, \]
\[ \frac{\partial u}{\partial \theta}(r, \pi/2, z) = 0, \quad \frac{\partial u}{\partial r}(a, \theta, z) = 0 \]

(c) \[ u(r, \theta, 0) = 0, \quad u(r, \theta, H) = 0, \quad \frac{\partial u}{\partial \theta}(r, 0, z) = 0, \]
\[ u(r, \pi/2, z) = 0, \quad \frac{\partial u}{\partial r}(a, \theta, z) = 0 \]

7.9.4. Solve the heat equation \[ \frac{\partial u}{\partial t} = k\nabla^2 u \]
inside a cylinder (of radius $a$ and height $H$) subject to the initial condition,

$$u(r, \theta, z, 0) = f(r, z),$$

independent of $\theta$, if the boundary conditions are

\begin{align*}
(a) \quad & u(r, \theta, 0, t) = 0, & u(r, \theta, H, t) = 0, & u(a, \theta, z, t) = 0 \\
(b) \quad & \frac{\partial u}{\partial z} (r, \theta, 0, t) = 0, & \frac{\partial u}{\partial z} (r, \theta, H, t) = 0, & \frac{\partial u}{\partial r} (a, \theta, z, t) = 0 \\
(c) \quad & u(r, \theta, 0, t) = 0, & u(r, \theta, H, t) = 0, & \frac{\partial u}{\partial r} (a, \theta, z, t) = 0
\end{align*}

7.9.5. Determine the three ordinary differential equations obtained by separation of variables for Laplace’s equation in spherical coordinates

$$0 = \frac{\partial}{\partial r} \left( r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial u}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2}.$$