are eigenvalue problems. In general, for a partial differential equation in $N$ variables that completely separates. there will be $N$ ordinary differential equations, $N-1$ of which are one-dimensional eigenvalue problems (to determine the $N-1$ separation constants). We have already shown this for $N = 3$ (this section) and $N = 2$.

## EXERCISES 7.3

7.3.1. Consider the heat equation in a two-dimensional rectangular region $0 < x < L, 0 < y < H$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

* (a) $u(0, y, t) = 0$, $u(L, y, t) = 0$, $u(x, 0, t) = 0$, $u(x, H, t) = 0$
(b) $\frac{\partial u}{\partial x}(0, y, t) = 0$, $\frac{\partial u}{\partial x}(L, y, t) = 0$, $\frac{\partial u}{\partial y}(x, 0, t) = 0$, $\frac{\partial u}{\partial y}(x, H, t) = 0$
* (c) $\frac{\partial u}{\partial x}(0, y, t) = 0$, $\frac{\partial u}{\partial x}(L, y, t) = 0$, $u(x, 0, t) = 0$, $u(x, H, t) = 0$
(d) $u(0, y, t) = 0$, $\frac{\partial u}{\partial x}(L, y, t) = 0$, $\frac{\partial u}{\partial y}(x, 0, t) = 0$, $\frac{\partial u}{\partial y}(x, H, t) = 0$
(e) $u(0, y, t) = 0$, $u(L, y, t) = 0$, $u(x, 0, t) = 0$,
$\frac{\partial u}{\partial y}(x, H, t) + hu(x, H, t) = 0$, $(h > 0)$

7.3.2. Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L$, $0 < y < H$, $0 < z < W$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

subject to the initial condition

$$u(x, y, z, 0) = f(x, y, z).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

(a) $u(0, y, z, t) = 0$, $\frac{\partial u}{\partial y}(x, 0, z, t) = 0$, $\frac{\partial u}{\partial z}(x, y, 0, t) = 0$,
$u(L, y, z, t) = 0$, $\frac{\partial u}{\partial y}(x, H, z, t) = 0$, $u(x, y, W, t) = 0$

* (b) $\frac{\partial u}{\partial x}(0, y, z, t) = 0$, $\frac{\partial u}{\partial y}(x, 0, z, t) = 0$, $\frac{\partial u}{\partial z}(x, y, 0, t) = 0$,
$\frac{\partial u}{\partial x}(L, y, z, t) = 0$, $\frac{\partial u}{\partial y}(x, H, z, t) = 0$, $\frac{\partial u}{\partial z}(x, y, W, t) = 0$

7.3.3 Solve

$$\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2}$$

on a rectangle $(0 < x < L, 0 < y < H)$ subject to

$$u(x,y,0) = f(x,y) \quad \begin{aligned} u(0,y,t) &= 0 \quad & \tfrac{\partial u}{\partial y}(x,0,t) &= 0 \\ u(L,y,t) &= 0 \quad & \tfrac{\partial u}{\partial y}(x,H,t) &= 0. \end{aligned}$$

7.3.4. Consider the wave equation for a vibrating rectangular membrane $(0 < x < L,\ 0 < y < H)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial conditions

$$u(x,y,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x,y,0) = f(x,y).$$

Solve the initial value problem if

(a) $u(0,y,t) = 0, \quad u(L,y,t) = 0, \quad \frac{\partial u}{\partial y}(x,0,t) = 0, \quad \frac{\partial u}{\partial y}(x,H,t) = 0$

*(b) $\frac{\partial u}{\partial x}(0,y,t) = 0, \quad \frac{\partial u}{\partial x}(L,y,t) = 0, \quad \frac{\partial u}{\partial y}(x,0,t) = 0, \quad \frac{\partial u}{\partial y}(x,H,t) = 0$

7.3.5. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - k\frac{\partial u}{\partial t} \quad \text{with } k > 0.$$

(a) Give a *brief* physical interpretation of this equation.

(b) Suppose that $u(x,y,t) = f(x)g(y)h(t)$. What ordinary differential equations are satisfied by $f$, $g$, and $h$?

7.3.6. Consider Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in a right cylinder whose base is arbitrarily shaped (see Fig. 7.3.3). The top is $z = H$ and the bottom is $z = 0$. Assume that

$$\begin{aligned} \tfrac{\partial}{\partial z} u(x,y,0) &= 0 \\ u(x,y,H) &= f(x,y) \end{aligned}$$

and $u = 0$ on the "lateral" sides.

(a) Separate the $z$-variable in general.

*(b) Solve for $u(x,y,z)$ if the region is a rectangular box, $0 < x < L, 0 < y < W, 0 < z < H$.

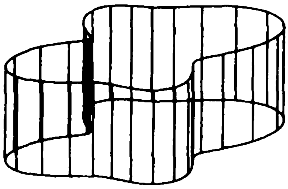

**Figure 7.3.3**

7.3.7. If possible, solve Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

in a rectangular-shaped region, $0 < x < L, 0 < y < W, 0 < z < H$, subject to the boundary conditions

| | | | |
|---|---|---|---|
| (a) | $\frac{\partial u}{\partial x}(0, y, z) = 0,$ | $u(x, 0, z) = 0,$ | $u(x, y, 0) = f(x, y)$ |
| | $\frac{\partial u}{\partial x}(L, y, z) = 0,$ | $u(x, W, z) = 0,$ | $u(x, y, H) = 0$ |
| (b) | $u(0, y, z) = 0,$ | $u(x, 0, z) = 0,$ | $u(x, y, 0) = 0,$ |
| | $u(L, y, z) = 0,$ | $u(x, W, z) = f(x, z),$ | $u(x, y, H) = 0$ |
| * (c) | $\frac{\partial u}{\partial x}(0, y, z) = 0,$ | $\frac{\partial u}{\partial y}(x, 0, z) = 0,$ | $\frac{\partial u}{\partial z}(x, y, 0) = 0$ |
| | $\frac{\partial u}{\partial x}(L, y, z) = f(y, z),$ | $\frac{\partial u}{\partial y}(x, W, z) = 0,$ | $\frac{\partial u}{\partial z}(x, y, H) = 0$ |
| * (d) | $\frac{\partial u}{\partial x}(0, y, z) = 0,$ | $\frac{\partial u}{\partial y}(x, 0, z) = 0,$ | $\frac{\partial u}{\partial z}(x, y, 0) = 0$ |
| | $u(L, y, z) = g(y, z),$ | $\frac{\partial u}{\partial y}(x, W, z) = 0,$ | $\frac{\partial u}{\partial z}(x, y, H) = 0$ |

## Appendix to 7.3: Outline of Alternative Method to Separate Variables

An alternative (and equivalent) method to separate variables for

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{7.3.33}$$

is to assume product solutions of the form

$$u(x, y, t) = f(x)g(y)h(t). \tag{7.3.34}$$

By substituting (7.3.34) into (7.3.33) and dividing by $c^2 f(x)g(y)h(t)$, we obtain

$$\frac{1}{c^2}\frac{1}{h}\frac{d^2 h}{dt^2} = \frac{1}{f}\frac{d^2 f}{dx^2} + \frac{1}{g}\frac{d^2 g}{dy^2} = -\lambda, \tag{7.3.35}$$