are eigenvalue problems. In general, for a partial differential equation in $N$ variables that completely separates, there will be $N$ ordinary differential equations, $N - 1$ of which are one-dimensional eigenvalue problems (to determine the $N - 1$ separation constants). We have already shown this for $N = 3$ (this section) and $N = 2$.

**EXERCISES 7.3**

7.3.1. Consider the heat equation in a two-dimensional rectangular region $0 < x < L, 0 < y < H$,

$$\frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

* (a) $u(0, y, t) = 0$, $u(L, y, t) = 0$, $u(x, 0, t) = 0$, $u(x, H, t) = 0$

(b) $\frac{\partial u}{\partial x}(0, y, t) = 0$, $\frac{\partial u}{\partial x}(L, y, t) = 0$, $u(x, 0, t) = 0$, $\frac{\partial u}{\partial y}(x, H, t) = 0$

* (c) $\frac{\partial u}{\partial x}(0, y, t) = 0$, $\frac{\partial u}{\partial x}(L, y, t) = 0$, $u(x, 0, t) = 0$, $u(x, H, t) = 0$

(d) $u(0, y, t) = 0$, $\frac{\partial u}{\partial x}(L, y, t) = 0$, $\frac{\partial u}{\partial y}(x, 0, t) = 0$, $\frac{\partial u}{\partial y}(x, H, t) = 0$

(e) $u(0, y, t) = 0$, $u(L, y, t) = 0$, $u(x, 0, t) = 0$, $\frac{\partial u}{\partial y}(x, H, t) + h u(x, H, t) = 0$. ($h > 0$)

7.3.2. Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L, 0 < y < H, 0 < z < W$,

$$\frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

subject to the initial condition

$$u(x, y, z, 0) = f(x, y, z).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

(a) $u(0, y, z, t) = 0$, $\frac{\partial u}{\partial y}(x, 0, z, t) = 0$, $\frac{\partial u}{\partial z}(x, y, 0, t) = 0$

$u(L, y, z, t) = 0$, $\frac{\partial u}{\partial y}(x, H, z, t) = 0$, $u(x, y, W, t) = 0$

* (b) $\frac{\partial u}{\partial z}(0, y, z, t) = 0$, $\frac{\partial u}{\partial y}(x, 0, z, t) = 0$, $\frac{\partial u}{\partial z}(x, y, 0, t) = 0$

$\frac{\partial u}{\partial z}(L, y, z, t) = 0$, $\frac{\partial u}{\partial y}(x, H, z, t) = 0$, $\frac{\partial u}{\partial z}(x, y, W, t) = 0$
7.3. Solve
\[ \frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2} \]
on a rectangle \((0 < x < L, 0 < y < H)\) subject to
\begin{align*}
    u(x, y, 0) &= f(x, y) \\
    u(0, y, t) &= 0 \\
    u(L, y, t) &= 0 \\
    \frac{\partial u}{\partial y}(x, 0, t) &= 0 \\
    \frac{\partial u}{\partial y}(x, H, t) &= 0.
\end{align*}

7.3.4. Consider the wave equation for a vibrating rectangular membrane \((0 < x < L, 0 < y < H)\)
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]
subject to the initial conditions
\begin{align*}
    u(x, y, 0) &= 0 \\
    \frac{\partial u}{\partial t}(x, y, 0) &= f(x, y).
\end{align*}
Solve the initial value problem if
\begin{enumerate}
    \item \(u(0, y, t) = 0, \ u(L, y, t) = 0, \ \frac{\partial u}{\partial y}(x, 0, t) = 0, \ \frac{\partial u}{\partial y}(x, H, t) = 0\)
    \item \(\frac{\partial u}{\partial x}(0, y, t) = 0, \ \frac{\partial u}{\partial x}(L, y, t) = 0, \ \frac{\partial u}{\partial y}(x, 0, t) = 0, \ \frac{\partial u}{\partial y}(x, H, t) = 0\)
\end{enumerate}

7.3.5. Consider
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - k \frac{\partial u}{\partial t} \]
with \(k > 0\).
\begin{enumerate}
    \item Give a brief physical interpretation of this equation.
    \item Suppose that \(u(x, y, t) = f(x)g(y)h(t)\). What ordinary differential equations are satisfied by \(f, g, \) and \(h?\)
\end{enumerate}

7.3.6. Consider Laplace's equation
\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \]
in a right cylinder whose base is arbitrarily shaped (see Fig. 7.3.3). The top is \(z = H\) and the bottom is \(z = 0\). Assume that
\begin{align*}
    \frac{\partial}{\partial z} u(x, y, 0) &= 0 \\
    u(x, y, H) &= f(x, y)
\end{align*}
and \(u = 0\) on the "lateral" sides.
\begin{enumerate}
    \item Separate the \(z\)-variable in general.
    \item Solve for \(u(x, y, z)\) if the region is a rectangular box, \(0 < x < L, 0 < y < W, 0 < z < H\).
7.3.7. If possible, solve Laplace's equation

\[ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, \]

in a rectangular-shaped region, \(0 < x < L, 0 < y < W, 0 < z < H\), subject to the boundary conditions

(a) \( \frac{\partial u}{\partial x}(0, y, z) = 0, \quad u(x, 0, z) = 0, \quad u(x, y, 0) = f(x, y) \)

(b) \( u(0, y, z) = 0, \quad u(x, 0, z) = 0, \quad u(x, y, 0) = 0, \quad u(x, W, z) = f(x, z) \)

* (c) \( \frac{\partial u}{\partial x}(L, y, z) = 0, \quad \frac{\partial u}{\partial y}(x, 0, z) = 0, \quad \frac{\partial u}{\partial z}(x, y, 0) = 0 \)

* (d) \( \frac{\partial u}{\partial x}(0, y, z) = 0, \quad \frac{\partial u}{\partial y}(x, 0, z) = 0, \quad \frac{\partial u}{\partial z}(x, y, 0) = 0 \)

\( u(L, y, z) = g(y, z), \quad \frac{\partial u}{\partial y}(x, W, z) = 0, \quad \frac{\partial u}{\partial z}(x, y, H) = 0 \)

Appendix to 7.3: Outline of Alternative Method to Separate Variables

An alternative (and equivalent) method to separate variables for

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

is to assume product solutions of the form

\[ u(x, y, t) = f(x)g(y)h(t). \]  

By substituting (7.3.34) into (7.3.33) and dividing by \( c^2 f(x)g(y)h(t) \), we obtain

\[ \frac{1}{c^2} \frac{d^2 h}{dt^2} = \frac{1}{f} \frac{d^2 f}{dx^2} + \frac{1}{g} \frac{d^2 g}{dy^2} = -\lambda, \]  

Figure 7.3.3