EXERCISES 5.3

*5.3.1. Do Exercise 4.4.2(b). Show that the partial differential equation may be put into Sturm-Liouville form.

5.3.2. Consider

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u + \beta\frac{\partial u}{\partial t}.$$

(a) Give a brief physical interpretation. What signs must α and β have to be physical?

(b) Allow ρ, α, β to be functions of x. Show that separation of variables works only if $\beta = c\rho$, where c is a constant.

(c) If $\beta = c\rho$, show that the spatial equation is a Sturm-Liouville differential equation. Solve the time equation.

*5.3.3. Consider the non-Sturm-Liouville differential equation

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

5.3.4. Consider heat flow with convection (see Exercise 1.5.2):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - V_0\frac{\partial u}{\partial x}.$$

(a) Show that the spatial ordinary differential equation obtained by separation of variables is not in Sturm-Liouville form.

*(b) Solve the initial boundary value problem

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

(c) Solve the initial boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0 \\ \frac{\partial u}{\partial x}(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

5.3.5. For the Sturm-Liouville eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0,$$

verify the following general properties:

(a) There is an infinite number of eigenvalues with a smallest but no largest.

(b) The nth eigenfunction has $n-1$ zeros.

(c) The eigenfunctions are complete and orthogonal.

(d) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

5.3.6. Redo Exercise 5.3.5 for the Sturm-Liouville eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

5.3.7. Which of statements 1–5 of the theorems of this section are valid for the following eigenvalue problem?

$$\begin{aligned} \frac{d^2\phi}{dx^2} + \lambda\phi &= 0 \quad \text{with} \\ \phi(-L) &= \phi(L) \\ \frac{d\phi}{dx}(-L) &= \frac{d\phi}{dx}(L). \end{aligned}$$

5.3.8. Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

5.3.9. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = 0, \quad \text{and} \quad \phi(b) = 0. \quad (5.3.10)$$

(a) Show that multiplying by $1/x$ puts this in the Sturm-Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)

(b) Show that $\lambda \geq 0$.

*(c) Since (5.3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.

(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

(e) Show that the nth eigenfunction has $n-1$ zeros.

5.3.10. Reconsider Exercise 5.3.9 with the boundary conditions

$$\frac{d\phi}{dx}(1) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(b) = 0.$$