[for a one-dimensional example, see Exercise 1.4.7(b)]. To show this, we integrate $\nabla^2 u = 0$ over the entire two-dimensional region

$$0 = \iint \nabla^2 u \; dx \; dy = \iint \nabla \cdot (\nabla u) \; dx \; dy.$$

Using the (two-dimensional) divergence theorem, we conclude that (see Exercise 1.5.8)

$$0 = \oint \nabla u \cdot \hat{n} \; ds. \tag{2.5.61}$$

Since $\nabla u \cdot \hat{n}$ is proportional to the heat flow through the boundary, (2.5.61) implies that the *net* heat flow through the boundary must be zero in order for a steady state to exist. This is clear physically, because otherwise there would be a change (in time) of the thermal energy inside, violating the steady-state assumption. Equation (2.5.61) is called the **solvability condition** or **compatibility condition** for Laplace's equation.

EXERCISES 2.5

2.5.1. Solve Laplace's equation inside a rectangle $0 \le x \le L,\ 0 \le y \le H$, with the following boundary conditions:

*(a) $\frac{\partial u}{\partial x}(0,y) = 0,\quad \frac{\partial u}{\partial x}(L,y) = 0,\quad u(x,0) = 0,\quad u(x,H) = f(x)$

(b) $\frac{\partial u}{\partial x}(0,y) = g(y),\quad \frac{\partial u}{\partial x}(L,y) = 0,\quad u(x,0) = 0,\quad u(x,H) = 0$

*(c) $\frac{\partial u}{\partial x}(0,y) = 0,\quad u(L,y) = g(y),\quad u(x,0) = 0,\quad u(x,H) = 0$

(d) $u(0,y) = g(y),\quad u(L,y) = 0,\quad \frac{\partial u}{\partial y}(x,0) = 0,\quad u(x,H) = 0$

*(e) $u(0,y) = 0,\quad u(L,y) = 0,\quad u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0,\quad u(x,H) = f(x)$

(f) $u(0,y) = f(y),\quad u(L,y) = 0,\quad \frac{\partial u}{\partial y}(x,0) = 0,\quad \frac{\partial u}{\partial y}(x,H) = 0$

(g) $\frac{\partial u}{\partial x}(0,y) = 0,\quad \frac{\partial u}{\partial x}(L,y) = 0,\quad u(x,0) = \begin{cases} 0 & x > L/2 \\ 1 & x < L/2 \end{cases},\quad \frac{\partial u}{\partial y}(x,H) = 0$

2.5.2. Consider $u(x,y)$ satisfying Laplace's equation inside a rectangle $(0 < x < L,\ 0 < y < H)$ subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0 \qquad \frac{\partial u}{\partial y}(x,0) = 0$$
$$\frac{\partial u}{\partial x}(L,y) = 0 \qquad \frac{\partial u}{\partial y}(x,H) = f(x).$$

*(a) *Without* solving this problem, briefly explain the physical condition under which there is a solution to this problem.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a).

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation (1.5.11) subject to the initial condition

$$u(x, y, 0) = g(x, y).$$

*2.5.3. Solve Laplace's equation *outside* a circular disk ($r \geq a$) subject to the boundary condition

(a) $u(a, \theta) = \ln 2 + 4 \cos 3\theta$

(b) $u(a, \theta) = f(\theta)$

You may assume that $u(r, \theta)$ remains finite as $r \to \infty$.

*2.5.4. For Laplace's equation inside a circular disk ($r \leq a$), using (2.5.45) and (2.5.47), show that

$$u(r, \theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\bar{\theta}) \left[-\frac{1}{2} + \sum_{n=0}^{\infty} \left(\frac{r}{a} \right)^n \cos n(\theta - \bar{\theta}) \right] d\bar{\theta}.$$

Using $\cos z = \text{Re}\,[e^{iz}]$, sum the resulting geometric series to obtain Poisson's integral formula.

2.5.5. Solve Laplace's equation inside the quarter-circle of radius 1 ($0 \leq \theta \leq \pi/2,\ 0 \leq r \leq 1$) subject to the boundary conditions

* (a) $\frac{\partial u}{\partial \theta}(r, 0) = 0, \quad u\left(r, \frac{\pi}{2}\right) = 0, \quad u(1, \theta) = f(\theta)$

(b) $\frac{\partial u}{\partial \theta}(r, 0) = 0, \quad \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{2}\right) = 0, \quad u(1, \theta) = f(\theta)$

* (c) $u(r, 0) = 0, \quad u\left(r, \frac{\pi}{2}\right) = 0, \quad \frac{\partial u}{\partial r}(1, \theta) = f(\theta)$

(d) $\frac{\partial u}{\partial \theta}(r, 0) = 0, \quad \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{2}\right) = 0, \quad \frac{\partial u}{\partial r}(1, \theta) = g(\theta)$

Show that the solution [part (d)] exists only if $\int_0^{\pi/2} g(\theta)\, d\theta = 0$. Explain this condition physically.

2.5.6. Solve Laplace's equation inside a semicircle of radius $a (0 < r < a,\ 0 < \theta < \pi)$ subject to the boundary conditions

*(a) $u = 0$ on the diameter and $u(a, \theta) = g(\theta)$

(b) the diameter is insulated and $u(a, \theta) = g(\theta)$

2.5.7. Solve Laplace's equation inside a 60° wedge of radius a subject to the boundary conditions

(a) $u(r, 0) = 0, \quad u\left(r, \frac{\pi}{3}\right) = 0, \quad u(a, \theta) = f(\theta)$

* (b) $\frac{\partial u}{\partial \theta}(r, 0) = 0, \quad \frac{\partial u}{\partial \theta}\left(r, \frac{\pi}{3}\right) = 0, \quad u(a, \theta) = f(\theta)$

2.5.8. Solve Laplace's equation inside a circular annulus ($a < r < b$) subject to the boundary conditions

* (a) $u(a, \theta) = f(\theta), \quad u(b, \theta) = g(\theta)$

(b) $\frac{\partial u}{\partial r}(a, \theta) = 0, \quad u(b, \theta) = g(\theta)$

(c) $\frac{\partial u}{\partial r}(a, \theta) = f(\theta), \quad \frac{\partial u}{\partial r}(b, \theta) = g(\theta)$

If there is a solvability condition, state it and explain it physically.

*2.5.9. Solve Laplace's equation inside a 90° sector of a circular annulus ($a < r < b$, $0 < \theta < \pi/2$) subject to the boundary conditions

(a) $u(r, 0) = 0, \quad u(r, \pi/2) = 0, \quad u(a, \theta) = 0, \quad u(b, \theta) = f(\theta)$

(b) $u(r, 0) = 0, \quad u(r, \pi/2) = f(r), \quad u(a, \theta) = 0, \quad u(b, \theta) = 0$

2.5.10. Using the maximum principles for Laplace's equation, prove that the solution of Poisson's equation, $\nabla^2 u = g(\boldsymbol{x})$, subject to $u = f(\boldsymbol{x})$ on the boundary, is unique.

2.5.11. Do Exercise 1.5.8.

2.5.12. (a) Using the divergence theorem, determine an alternative expression for $\iint u \, \nabla^2 u \, dx \, dy \, dz$.

(b) Using part (a), prove that the solution of Laplace's equation $\nabla^2 u = 0$ (with u given on the boundary) is unique.

(c) Modify part (b) if $\nabla u \cdot \hat{\boldsymbol{n}} = 0$ on the boundary.

(d) Modify part (b) if $\nabla u \cdot \hat{\boldsymbol{n}} + hu = 0$ on the boundary. Show that Newton's law of cooling corresponds to $h < 0$.

2.5.13. Prove that the temperature satisfying Laplace's equation cannot attain its minimum in the interior.

2.5.14. Show that the "backward" heat equation

$$\frac{\partial u}{\partial t} = -k \frac{\partial^2 u}{\partial x^2},$$

subject to $u(0, t) = u(L, t) = 0$ and $u(x, 0) = f(x)$, is *not* well posed. [*Hint:* Show that if the data are changed an arbitrarily small amount, for example,

$$f(x) \to f(x) + \frac{1}{n} \sin \frac{n\pi x}{L}$$

for large n, then the solution $u(x, t)$ changes by a large amount.]

2.5.15. Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty$, $0 < y < H$) subject to the boundary conditions

(a) $\frac{\partial u}{\partial y}(x,0) = 0$, $\frac{\partial u}{\partial y}(x,H) = 0$, $u(0,y) = f(y)$

(b) $u(x,0) = 0$, $u(x,H) = 0$, $u(0,y) = f(y)$

(c) $u(x,0) = 0$, $u(x,H) = 0$, $\frac{\partial u}{\partial x}(0,y) = f(y)$

(d) $\frac{\partial u}{\partial y}(x,0) = 0$, $\frac{\partial u}{\partial y}(x,H) = 0$, $\frac{\partial u}{\partial x}(0,y) = f(y)$

Show that the solution [part (d)] exists only if $\int_0^H f(y)\,dy = 0$.

2.5.16. Consider Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad \frac{\partial u}{\partial x}(L,y) = g(y), \quad \frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,H) = f(x).$$

(a) What is the solvability condition and its physical interpretation?

(b) Show that $u(x,y) = A(x^2 - y^2)$ is a solution if $f(x)$ and $g(y)$ are constants [under the conditions of part (a)].

(c) Under the conditions of part (a), solve the general case [nonconstant $f(x)$ and $g(y)$]. [*Hints:* Use part (b) and the fact that $f(x) = f_{\text{av}} + [f(x) - f_{\text{av}}]$, where $f_{\text{av}} = \frac{1}{L}\int_0^L f(x)\,dx$.]

2.5.17. Show that the mass density $\rho(x,t)$ satisfies $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$ due to conservation of mass.

2.5.18. If the mass density is constant, using the result of Exercise 2.5.17, show that $\nabla \cdot \boldsymbol{u} = 0$.

2.5.19. Show that the streamlines are parallel to the fluid velocity.

2.5.20. Show that anytime there is a stream function, $\nabla \times \boldsymbol{u} = 0$.

2.5.21. From $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$, derive $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $u_\theta = -\frac{\partial \psi}{\partial r}$.

2.5.22. Show the drag force is zero for a uniform flow past a cylinder including circulation.

2.5.23. Consider the velocity u_θ at the cylinder. Where do the maximum and minimum occur?

2.5.24. Consider the velocity u_θ at the cylinder. If the circulation is negative, show that the velocity will be larger above the cylinder than below.

2.5.25. A stagnation point is a place where $\boldsymbol{u} = 0$. For what values of the circulation does a stagnation point exist on the cylinder?

2.5.26. For what values of θ will $u_r = 0$ off the cylinder? For these θ, where (for what values of r) will $u_\theta = 0$ also?

2.5.27. Show that $\psi = \alpha \frac{\sin\theta}{r}$ satisfies Laplace's equation. Show that the streamlines are circles. Graph the streamlines.