EXERCISES 2.3

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

\( \frac{\partial u}{\partial t} = k \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \)  
\( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \)  
\( \frac{\partial u}{\partial t} = k \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) \)  
\( \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \)

2.3.2. Consider the differential equation

\[ \frac{d^2 \phi}{dx^2} + \lambda \phi = 0. \]

Determine the eigenvalues \( \lambda \) (and corresponding eigenfunctions) if \( \phi \) satisfies the following boundary conditions. Analyze three cases (\( \lambda > 0, \lambda = 0, \lambda < 0 \)). You may assume that the eigenvalues are real.

(a) \( \phi(0) = 0 \) and \( \phi(\pi) = 0 \)

(b) \( \phi(0) = 0 \) and \( \phi(1) = 0 \)

(c) \( \frac{d\phi}{dx}(0) = 0 \) and \( \frac{d\phi}{dx}(L) = 0 \) (If necessary, see Sec. 2.4.1.)

(d) \( \phi(0) = 0 \) and \( \frac{d\phi}{dx}(L) = 0 \)

(e) \( \frac{d\phi}{dx}(0) = 0 \) and \( \phi(L) = 0 \)

(f) \( \phi(a) = 0 \) and \( \phi(b) = 0 \) (You may assume that \( \lambda > 0 \).)

(g) \( \phi(0) = 0 \) and \( \frac{d\phi}{dx}(L) + \phi(L) = 0 \) (If necessary, see Sec. 5.8.)

2.3.3. Consider the heat equation

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \]

subject to the boundary conditions

\( u(0, t) = 0 \) and \( u(L, t) = 0. \)

Solve the initial value problem if the temperature is initially

(a) \( u(x, 0) = 6 \sin \frac{9\pi x}{L} \)  
(b) \( u(x, 0) = 3 \sin \frac{\pi x}{L} - \sin \frac{3\pi x}{L} \)

(c) \( u(x, 0) = 2 \cos \frac{3\pi x}{L} \)  
(d) \( u(x, 0) = \begin{cases} 1 & 0 < x \leq L/2 \\ 2 & L/2 < x < L \end{cases} \)
2.3.4. Consider
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \]
subject to \( u(0, t) = 0, u(L, t) = 0, \) and \( u(x, 0) = f(x). \)

*(a) What is the total heat energy in the rod as a function of time?
(b) What is the flow of heat energy out of the rod at \( x = 0 \)? at \( x = L \)?
*(c) What relationship should exist between parts (a) and (b)?

2.3.5. Evaluate (be careful if \( n = m \))
\[ \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx \quad \text{for } n > 0, m > 0. \]

Use the trigonometric identity
\[ \sin a \sin b = \frac{1}{2} [\cos(a - b) - \cos(a + b)]. \]

*2.3.6. Evaluate
\[ \int_0^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx \quad \text{for } n \geq 0, m \geq 0. \]

Use the trigonometric identity
\[ \cos a \cos b = \frac{1}{2} [\cos(a + b) + \cos(a - b)]. \]

(\text{Be careful if } a - b = 0 \text{ or } a + b = 0.)

2.3.7. Consider the following boundary value problem (if necessary, see Sec. 2.4.1):
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0, t) = 0, \frac{\partial u}{\partial x}(L, t) = 0, \quad \text{and} \quad u(x, 0) = f(x). \]

(a) Give a one-sentence physical interpretation of this problem.
(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [\text{Hint: The answer is}]

\[ u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n kt} \cos \frac{n\pi x}{L}. \]

What is \( \lambda_n \)?
2.3. Heat Equation With Zero Temperature Ends

(c) Show that the initial condition, \( u(x, 0) = f(x) \), is satisfied if
\[
f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.
\]

(d) Using Exercise 2.3.6, solve for \( A_0 \) and \( A_n (n \geq 1) \).

(e) What happens to the temperature distribution as \( t \to \infty \)? Show that it approaches the steady-state temperature distribution (see Sec. 1.4).

*2.3.8. Consider
\[
\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} - \alpha u.
\]
This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° (\( \alpha > 0 \), see Exercise 1.2.4) or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are
\[
u(0, t) = 0 \quad \text{and} \quad u(L, t) = 0.
\]

(a) What are the possible equilibrium temperature distributions if \( \alpha > 0 \)?

(b) Solve the time-dependent problem \([u(x, 0) = f(x)]\) if \( \alpha > 0 \). Analyze the temperature for large time \((t \to \infty)\) and compare to part (a).

*2.3.9. Redo Exercise 2.3.8 if \( \alpha < 0 \). [Be especially careful if \(-\alpha/k = (n\pi/L)^2\).]

2.3.10. For two- and three-dimensional vectors, the fundamental property of dot products, \( A \cdot B = |A||B| \cos \theta \), implies that
\[
|A \cdot B| \leq |A||B|.
\]  
(2.3.44)

In this exercise we generalize this to \( n \)-dimensional vectors and functions, in which case (2.3.44) is known as **Schwarz’s inequality**. [The names of Cauchy and Buniakovsky are also associated with (2.3.44).]

(a) Show that \(|A - \gamma B|^2 > 0\) implies (2.3.44), where \( \gamma = A \cdot B/B \cdot B \).

(b) Express the inequality using both
\[
A \cdot B = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.
\]

*(c) Generalize (2.3.44) to functions. [Hint: Let \( A \cdot B \) mean the integral \( \int_0^L A(x)B(x) \, dx \).]

2.3.11. Solve Laplace’s equation inside a rectangle:
\[
\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]
subject to the boundary conditions
\[
\begin{align*}
u(0, y) &= g(y) & u(x, 0) &= 0 \\
u(L, y) &= 0 & u(x, H) &= 0.
\end{align*}
\]

(Hint: If necessary, see Sec. 2.5.1.)