Use l'Hôpital's rule to evaluate

\[ \lim_{x \to 0} \frac{1 + x - \sqrt{1 + 2x}}{x^2} \]

**ANSWER**

\[
\lim_{x \to 0} \frac{1 + x - \sqrt{1 + 2x}}{x^2} = \frac{1 - 1}{0} = \frac{0}{0}
\]

L.H. \[= \lim_{x \to 0} \frac{1 - \frac{1}{\sqrt{1 + 2x}}}{2x} = \frac{0}{0}\]

L.H. \[= \lim_{x \to 0} \frac{\frac{1}{2} (1 + 2x)^{-3/2} \cdot 2}{2}
\]

\[= \lim_{x \to 0} \frac{(1 + 2x)^{-3/2}}{2}
\]

\[= \frac{(1 + 2 \cdot 0)^{-3/2}}{2}
\]

\[= \frac{1}{2}\]
QUESTION 2

Use calculus and optimization to find the maximum area of a rectangle that can be contained inside a circle of radius 1. Justify that the value you find is a maximum.

ANSWER

Let \( x \) be the length of the rectangle, and \( y \) be the height.

The diagonal is a diameter of the circle so
\[
x^2 + y^2 = 2^2
\]
\[
x = \pm \sqrt{4-y^2}
\]
\[
x = +\sqrt{4-y^2}
\]
(because \( x \geq 0 \))

\[
A = xy = y\sqrt{4-y^2}
\]

\[
A' = \sqrt{4-y^2} + \frac{-y^2}{\sqrt{4-y^2}}
\]
\[
= \frac{4-2y^2}{\sqrt{4-y^2}}
\]

Critical Points:
\[
A' = 0 \text{ when } 4-2y^2 = 0
\]
\[
y = \pm \sqrt{2}
\]
\[
y = +\sqrt{2}
\]

\[A' \text{ undefined at } y=0\]
\[
y = 2
\]

(over)
2 cont)

\[ A(0) = 0 \]
\[ A(2) = 0 \]
\[ A(\sqrt{2}) = 2 \]

\( A(y) \) is continuous, so by the Extreme Value Theorem, \( A(\sqrt{2}) = 2 \) is the global maximum.
QUESTION 3

For the function

\[ y = f(x) = \sqrt{4 - x^2} \]

(a) What is the domain of \( f(x) \)? Is \( f(x) \) even, odd, or neither? Find the intercepts of \( y = f(x) \). [10 points]

(b) Find and simplify the first and second derivatives of \( f(x) \). [10 points]

(c) Find the critical points of \( f(x) \). Where is the function increasing/decreasing? [8 points]

(d) If it applies, use the second derivative test to determine if the critical points are local max/mins. Which ones are also global extrema, if any? [4 points]

(e) Where is \( f(x) \) concave up/down? Are there any inflection points? If so, find them. [8 points]

(f) Use all the above information to sketch the graph of \( y = f(x) \). [10 points]

ANSWER

(a) Domain: \( 4 - x^2 \geq 0 \)
   \[(2-x)(2+x) \geq 0 \]
   \([-2, 2]\)

y-int.: Set \( x = 0 \)
\[ y = \sqrt{4} = 2 \]

f(-x) = \sqrt{4 - (-x)^2} = \sqrt{4 - x^2} = f(x)

Even

x-int.: Set \( y = 0 \)
\[ 0 = \sqrt{4 - x^2} \]
\[ 0 = 4 - x^2 \]
\[ 0 = (2-x)(2+x) \]
\[ x = 2 \quad x = -2 \]

(continue on the next page if you need more space)
(b) \[ y' = \frac{-x}{\sqrt{4-x^2}} \]

\[ y'' = \frac{-\sqrt{4-x^2} + x \cdot \frac{-x}{\sqrt{4-x^2}}}{4-x^2} = \frac{- (4-x^2) - x^2}{\sqrt{4-x^2} \cdot 4-x^2} \]

\[ = -\frac{4}{(4-x^2)^{3/2}} \]

(c) \[ y' = 0 \text{ when } x = 0 \]

\[ y' \text{ undefined at } x = \pm 2 \]

Critical Points:
\[ x = -2, 0, 2 \]

Increasing \([-2, 0]\)

Decreasing \([0, 2]\)

(continue on the next page if you need more space)
(d) \[ f''(-2) = \text{DNE} \Rightarrow 2^{\text{nd}} \text{ derivative test does not apply} \]
\[ f''(2) = \text{DNE} \]
\[ f'''(0) = -\frac{1}{2} \leq \text{local max} \]
Global max at \( x = 0 \) because of increasing/
Global min at \( x = \pm 2 \) decreasing information

(e) \( y'' = 0 \) never
\( y'' \) undefined at \( x = \pm 2 \)

Concave Down on \([-2, 2]\)

No points of inflection.

(f) \[ y = \sqrt{4-x^2} \]

\[ y \]
\[ x \]

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