

BUILDING MODELS UNDER CONSTRAINTS

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1. INTRODUCTION

This topic expositis and explores how properties of a theory affect the structural variation in its models. Our journey begins in section 2 with Morley’s Theorem, which states that if a complete countable theory is sufficiently constrained so that it only has one model of power $\aleph_1$, then it only has one model of every uncountable cardinality. Looking deeper at the overall structure of uncountably categorical theories, and in particular at their countable models, we arrive at the Baldwin–Lachlan Theorem: if an uncountably categorical theory has more than one countable model, it has countably many; and moreover, they “fall into a line” in a neat way. This new understanding of models of uncountably categorical theories showed that they are deeply connected to strongly minimal structures, on which there is a meaningful notion of geometry. An influential conjecture of Zil’ber stated that any such geometry falls into one of three classes. In the early 1990’s, however, Hrushovski developed a new and powerful technique for building new models that gave examples of geometries outside this trichotomy, as discussed in section 4. This new way of realising countable structures as limits of collections of finite structures not only settled Zil’ber’s conjecture and hence gave a deeper model-theoretic understanding of the Baldwin–Lachlan theorem, it also gave computability theorists a new tool for building objects called computable spectra, and thus gave a deeper computability-theoretic understanding of the Baldwin–Lachlan theorem. This is covered in section 5. Finally, section 5 also looks at how a model theoretic construction sheds computability theoretic light on algebraic structures.

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One recurring theme is model-building and this note touches on a number of different, but related, ways in which this is done. In particular, the shadow of set-theoretic forcing can be seen at multiple junctures. For example, Morley uses an argument reminiscent of forcing to build models prime over a given subset in certain theories. Forcing is also related to what is called a finite-injury priority argument in section 3.3. Indeed, both methods have roots in computability constructions like the finite extension method. (For more information, see [Soare, 1987].) Finally, as discussed in 5.2, model theoretic forcing, as the name suggests, is based on its set theoretic counterpart. The Hrushovski construction, covered in section 4.2, on the other hand, essentially carefully glues together finite structures to build countable structures with certain properties. Section 5 explores how both of these ideas continue to inform and direct current research in model theory and computability theory.

2. LAYING THE GROUNDWORK: MORLEY'S THEOREM

Throughout this text, T will be a complete countable theory with an infinite model, and all models will be infinite, unless otherwise stated. While we have tried to make this note as self-contained as possible, we acknowledge that a full introduction to model theory would not be possible in such short space. For a deeper background in the requisite model theory, see, for example, [Chang and Keisler, 1990].

The Löwenheim–Skolem Theorem states that if a countable theory T has an infinite model, then it has a model of every infinite cardinality. Thus, one cannot capture infinite cardinals with first order logic. Instead, the best one can hope for is that specifying a theory T and an infinite cardinal κ determines a model up to isomorphism. In this case, we say T is *categorical in power κ* , or *κ -categorical* for short.

Morley's Theorem was one of the first results that gave a general understanding of the restrictions that categoricity places on a theory. One has, for example, theories like the theory of pure identity, which is trivially categorical in every infinite cardinality. The theory of dense linear orders without endpoints, on the other hand, is $\aleph_0$ -categorical, but has 2^κ models of every uncountable cardinality κ . There are also theories like that of algebraically closed fields of characteristic 0, ACF_0 , which have countably many countable models, but only one up to isomorphism in every uncountable cardinality. Moreover, there are theories which are not categorical in any infinite cardinality, like the theory of $(\mathbb{N}, <)$.

Łoś conjectured that any countable theory which is categorical in some uncountable power must be categorical in every uncountable power. Michael Morley laid out a proof of this in 1962 in a groundbreaking University of Chicago PhD thesis, published as [Morley, 1965].

Theorem 2.0.1 (Morley, 1965). *Let T be a complete theory in a countable language and suppose T is κ -categorical for some uncountable κ . Then T is κ -categorical for every uncountable κ .*

The proof of this theorem is a *tour de force*, to which this note cannot hope to do justice. Instead, we highlight some of the key features of Morley's argument.

Morley's opening move was to define a notion, now known as *Morley rank*, of the complexity of an object called a *type*. It is reminiscent of the Cantor–Bendixson rank in general topology.

Definition 2.0.2. Let M be a structure for a language L .

An *n -type* (with respect to M) is a maximal consistent set of L -formulas in n free variables, which are consistent with the theory of M . Unless otherwise specified, a *type* is a *1-type*.

If $X \subseteq M$, then an n -type with parameters from X is an n -type in L_X , the language given by expanding L by adding a new constant for each element of X , consistent with the theory of M_X , the model formed by interpreting each new constant as the corresponding element of X .

Informally, an n -type Γ describes an n -tuple that “could” exist in M ; i.e., there is a model N with the same theory as M and extending M in the strong sense below, and an n -tuple of elements of N satisfying Γ .

Definition 2.0.3. N is an *elementary extension* of M , written $M \preceq N$ if and only if, for any formula ϕ and any n -tuple $\bar{m}$ of M ,

$$N \models \phi(\bar{m}) \quad \text{iff} \quad M \models \phi(\bar{m}).$$

Thus, not only do M and N satisfy the same sentences in L , but they agree on statements involving elements of the smaller model.

Definition 2.0.4 (Morley rank). Let M be a model of some theory T . Define $S^0(M) = S(M)$ to be the collection of 1-types with parameters from M , called the *Stone space* corresponding to M . Set $\text{Tr}^0(M)$ to be the collection of all 1-types that are satisfied by finitely many elements of M . Define:

- (i) Set $S^i(M) = S^{i-1}(M) \setminus \text{Tr}^{i-1}(M)$ for a successor ordinal i . For a limit ordinal $\alpha \neq 0$, set $S^\alpha(M) = \bigcap_{i < \alpha} S^i(M)$
- (ii) Set $\text{Tr}^i(M)$ to be the collection of types in $S^i(M)$ for which their extensions in any $N \succ M$ are all isolated points in $S^i(N)$

The type p has *transcendence* or *Morley rank* greater than α if $p \in S^{\alpha+1}$, and Morley rank equal to α if p has Morley rank greater than β for $\beta < \alpha$ but not greater than α . If p has Morley rank greater than every ordinal, then p is said to have rank ∞ .

Note that this is not the same as having Morley rank equal to some infinite ordinal.

A theory T is *totally transcendental* if every type consistent with T has an ordinal Morley rank.

We remark as an aside that the Stone space of the previous definition is named for Marshall Harvey Stone, who was known not only for his considerable personal contributions to mathematics, but also for his indirect contributions as chair of the University of Chicago Mathematics Department during its famous “Stone Age”. This was in the postwar period during the 1940’s and 1950’s.

An example of a totally transcendental theory is that of a single equivalence relation E with infinitely many infinite classes. Its types are of the form:

- (i) Types q_a generated by $\phi_a(x) = “x = a”$ for some $a \in M$
- (ii) Types of the form $p_a(x) = \{x \neq b \mid b \in M\} \cup \{E(x, a)\}$ for some $a \in M$
- (iii) A type of the form $p(x) = \{x \neq b \mid b \in M\} \cup \{\neg E(x, a) \mid a \in M\}$

These have Morley ranks 0, 1, and 2 respectively.

Similarly, ACF_0 is totally transcendental. Its types fall into the following three classes:

- (i) Types p_a generated by $\phi_a(x) = “x = a”$ for some $a \in M$
- (ii) Types p_P generated by $\phi_P(x) = “P(x) = 0”$ for some polynomial P
- (iii) A type p of the form $p(x) = \{\neg P(x) = 0 \mid P \text{ is a polynomial}\}$

These correspond to Morley rank 0, 0, and 1 respectively.

However, being totally transcendental is a strong property and there are many theories, even $\aleph_0$ -categorical ones, which do not satisfy it. For example, the type

$$p(x) = \{“x \neq a” \text{ for all } a \text{ and } “a_0 < x < a_1” \text{ for some } a_0, a_1\}$$

in the theory of dense linear orders has Morley rank ∞ .

The following theorem characterises being totally transcendental in terms of the number of types.

Theorem 2.0.5. *T is totally transcendental iff for every model M of T , and every countable $X \subseteq M$, the number of types over X is countable.*

Note that having countably many types over any countable set X is as small as one could hope for in this context. In particular, for every $a \in X$, the formula $x = a$ defines a unique type over X . This property of having exactly κ types over any set of size κ is known as κ -stability. While we will not discuss it further here, this notion has been incredibly fruitful in modern model theory as a way of distinguishing “tame” and “wild” theories.

The following theorem is at the centre of Morley’s proof and illustrates the relationship between categoricity and Morley rank.

Theorem 2.0.6. *Let T be a theory which is κ -categorical for some uncountable cardinal κ . Then T is totally transcendental.*

Thus, Morley’s analysis shows that the space of types of a theory which is categorical in some uncountable cardinal must have more structure than those of general theories. This is in some sense captured by every type having ordinal Morley rank.

However, the converse of Theorem 2.0.6 is not true. The theory of an equivalence relation with infinitely many infinite equivalence classes is an example of a theory that is totally transcendental, as shown above, but not κ -categorical for any uncountable κ .

It is worth noting that the proof of Theorem 2.0.6 requires using a construction known as Ehrenfeucht–Mostowski models to obtain models of size $\lambda \geq \aleph_1$ realising at most countably many types over a countable subset of the domain. This is an important model-theoretic construction that in some sense builds up a model in a bare-bones way over an *indiscernible sequence* (see [Chang and Keisler, 1990] for a definition).

On the other hand, if T is not totally transcendental, Theorem 2.0.5 implies the existence of a model N and a countable $X \subseteq N$ such that the number of types over X is uncountable. From this, one may obtain by standard arguments a model M of cardinality κ containing X , which realises uncountably many types over X . This cannot be isomorphic to the model of cardinality κ given by the Ehrenfeucht–Mostowski construction in the previous paragraph, so T cannot be κ -categorical.

We need one more property of totally transcendental theories in order to prove the main result.

Definition 2.0.7. A model M is *saturated* if it realises all types over sets $X \subseteq M$ of cardinality strictly less than that of M .

Note that we could not require that M realises all types over a set of size M . For example, the set of sentences $\{x \neq m : m \in M\}$ gives a type over the universe of M that cannot possibly be realised in M itself.

Informally, saturation stipulates that M contains as many kinds of elements as possible. Any element that can be “finitely approximated” using a set of parameters $X \subseteq M$ with $|X| < |M|$ must be realised in M .

Proposition 2.0.8. *Let T be categorical in some uncountable κ and M an uncountable model of T of any cardinality. Then M is saturated.*

The proof of this proposition requires several steps and ties together the threads of categoricity, being totally transcendental, and saturation, as well as a number

of other key concepts, including the existence of models prime over a set and the existence of large indiscernible sets, both for totally transcendental theories.

Now, we are in a position to prove the main theorem:

Theorem 2.0.9. *Let T be κ -categorical for some uncountable cardinal κ . Then T is categorical in every uncountable cardinal.*

Proof. Let T satisfy the hypothesis of the theorem. Then, T is totally transcendental. Hence, every uncountable model of T is saturated. Then a back-and-forth argument gives that any two uncountable, and hence saturated, models of the same cardinality are isomorphic, as required. $\square$

3. THE COUNTABLE CASE

Morley’s breakthrough on uncountable categoricity gave little information about what happens in the countable case. Morley did show in [?] that an uncountably categorical theory can have at most countably many models and it was conjectured by Vaught that an uncountably categorical theory must have either one or countably many models. This was resolved positively by Baldwin and Lachlan in 1971. Moreover, their result shows that in the not $\aleph_0$ -categorical case, the countable models have a lot of structure and line up in a neat way. On the other side of this dichotomy, Zil’ber showed that the $\aleph_0$ -categorical strongly minimal sets are Cherlin, Harrington, and Lachlan showed that in the totally categorical case, models of the theory can be coordinatised in a precise way over a “small” set.

3.1. The Baldwin–Lachlan Theorem. Clearly, both situations in Vaught’s conjecture are possible. The theory of pure identity is κ -categorical for every cardinal κ . On the other hand, the theory ACF_0 of algebraically closed fields of characteristic 0 is uncountably categorical, but has $\aleph_0$ countable models – one for each transcendence degree in $\{0, 1, \dots\} \cup \{\omega\}$.

What Baldwin and Lachlan showed in 1971 was that not only is it true that any $\aleph_1$ -categorical theory has one or countably many countable models, but moreover that theories in the latter case behave remarkably like ACF_0 in a sense that we will explain below.

A central concept in their result is the notion of strong minimality.

Definition 3.1.1. Let M be an $\mathcal{L}$ -structure. A definable subset $\phi[M] \subseteq M$ is *strongly minimal* if there exists k such that for any elementary extension $N \succ M$ and any definable with parameters $Y \subseteq N$, either $|\phi[N] \cap Y| \leq k$ or $|\phi[N] \setminus Y| \leq k$.

An $\mathcal{L}$ -formula in one free variable is *strongly minimal* if and only if it defines a strongly minimal set on some model of T .

As an example, consider any model of ACF_0 . Then it can be shown that the definable sets are boolean combinations of solution sets of polynomials. Thus, every definable set is finite or cofinite and hence the universe of the model itself is strongly minimal.

Similarly, it can be shown that an infinite-dimensional vector space over a finite field has strongly minimal universe.

Definition 3.1.2. A model M of T is *prime* if it admits an elementary embedding into any other model of T of the same cardinality.

In our setting (T complete, countable with an infinite model), when M is countable, this is equivalent to M only realising *principal types*; i.e., types that are determined by a single formula.

Note that this is in some sense dual to the notion of saturation. Every model of T must realise the principal types, so a prime model realises as few types as possible.

Theorem 3.1.3 (Baldwin–Lachlan, 1971). *Let T be a complete theory which is $\aleph_1$ - but not $\aleph_0$ -categorical. Then T has countably many countable models and moreover, the countable models can be arranged in an $\omega + 1$ chain so that*

$$M_0 \prec M_1 \prec M_2 \prec \cdots \prec M_\omega$$

where M_0 is prime, M_ω is saturated, and each M_i is uniquely determined by its dimension over a strongly minimal set in some principal extension.

The standard examples of $\aleph_1$ - but not $\aleph_0$ -categorical theories are ACF_0 , as discussed above, and the theory of a vector space over a countable field. In this case, the countable models are the finite and countably-infinite dimensional spaces.

We will now overview the proof of the Baldwin–Lachlan result. It had been proven earlier by Marsh [Marsh, 1966] that any uncountably categorical theory with more than one countable model has countably many countable models when the models in question have a strongly minimal universe. Substantially generalising this, Baldwin and Lachlan showed that any model of an $\aleph_1$ -categorical theory is in some way built from a strongly minimal one, thus allowing Marsh’s results to be extended to full generality.

Nonetheless, being uncountably categorical does not imply that every model of T is strongly minimal, or even the algebraic closure of a strongly minimal set. The Lachlan counterexample construction can be used to build such a model, see e.g. [Baldwin and Lachlan, 1971], §4.

In analogy with linear algebra, we make the following definitions.

Definition 3.1.4. For a model M and subset $X \subseteq M$, define the *closure* of X to be

$$\text{cl}(X) = \bigcup \{Y : Y \text{ is definable with parameters from } X \text{ and } Y \text{ is finite}\}.$$

Definition 3.1.5. Say a subset $X \subseteq M$ is *independent* if any $X' \subsetneq X$ has $\text{cl}(X') \subsetneq \text{cl}(X)$. $Y \subseteq X$ is a *basis* for X if it is independent and $\text{cl}(Y) = X$. Set $\dim(X)$ to be the cardinality of any basis. Dimension is well defined when X is a subset of a strongly minimal structure.

We will sketch the proof of Marsh’s result here. Define a *prime extension* of a model M to be any $N \succ M$, $M \neq N$, such that any elementary embedding $f : M \rightarrow M'$ factors through N . It follows from a result called Vaught’s two cardinal theorem that if T is $\aleph_1$ -categorical and $M \prec N$, $M \neq N$, then, for any formula ϕ , $\phi[M] \subsetneq \phi[N]$. As shown in [Morley, 1965], if T is $\aleph_1$ -categorical, then any model M of T has a minimal prime extension and, moreover, that extension is unique up to isomorphism. From this one can show that an $\aleph_1$ -categorical theory T has at most countably many models. Finally, we also need the observation that the proof of Theorem 5.4 in [Morley, 1965] shows that if M is a non-saturated model of an $\aleph_1$ -categorical theory, then M cannot contain an infinite indiscernible subset; i.e., an infinite subset $X \subseteq M$ such that, for every n , any two n -tuples of distinct elements in X have the same n -type. Note that any basis is necessarily indiscernible.

Let T be an $\aleph_1$ - but not $\aleph_0$ -categorical theory. Let M_0 denote a prime model of T , which exists because T is uncountably categorical [Morley, 1965]. Because T is not $\aleph_0$ -categorical, M_0 is not saturated and hence cannot contain an infinite indiscernible subset. Thus, $\dim \phi[M_0] < \omega$, say k . Let M_i be the minimal prime extension of M_{i-1} and $M_\omega = \bigcup_{i < \omega} M_i$. Using the fact that $\phi[M_i] \subsetneq \phi[M_{i+1}]$ Marsh shows that $\dim(\text{cl} \phi[M_i]) = k + i$. Furthermore, an analysis of the theory of $(M_i, x_0, \dots, x_{k+i-1})$, where $x_{k+j} \in M_j$ and $x_0, \dots, x_{k+i-1}$ forms a basis for $\text{cl} \phi[M_i]$ shows that any countable N with $\dim(\text{cl} \phi[N]) = k + i$ is isomorphic to M_i , which ends the proof.

The following theorem, in the form stated here, is due to Baldwin and Lachlan, who recognised that any $\aleph_1$ - but not $\aleph_0$ -categorical theory can be expanded in a simple way to one which admits a strongly minimal set. A similar result appears at the end of Marsh's thesis.

Theorem 3.1.6. *Let T be $\aleph_1$ - but not $\aleph_0$ -categorical. Then T admits a principal extension T' , such that T' has a strongly minimal set.*

To remove the “strongly minimal” hypothesis from Marsh's result required an involved and technical argument, which we will not go into here. However, we will remark that this was a significant extension of Marsh's work.

3.2. Geometric structure. The proof of the Baldwin–Lachlan theorem (Theorem 3.1.3) suggests that to understand the models of uncountably categorical theories, we need to understand the structure of the strongly minimal set over which the countable models are built. As Marsh showed [Marsh, 1966], any strongly minimal set carries a *pregeometry* in the following sense.

Definition 3.2.1. A *pregeometry* is a set X with a closure operation $\text{cl} : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfying:

- (i) $\text{cl}(\text{cl}(A)) = \text{cl}(A)$
- (ii) If $A \subseteq B$, then $\text{cl}(A) \subseteq \text{cl}(B)$
- (iii) If $x \in \text{cl}(A \cup \{y\})$ then $y \in \text{cl}(A \cup \{x\})$ or $x \in A$
- (iv) If $x \in \text{cl}(A)$, then there is some finite $A' \subseteq A$ such that $x \in \text{cl}(A')$.

X is a *geometry* if, in addition, it satisfies that $\text{cl}(\{x\}) = \{x\}$ and $\text{cl}(\emptyset) = \emptyset$.

Zil'ber conjectured in [Zil'ber, a] that there are in fact only three types of pregeometry in this case.

Conjecture 3.2.2 (Zil'ber trichotomy conjecture). *Let M be a model with a strongly minimal universe. Then the pregeometry of M is isomorphic to something in one of the following classes of examples:*

- (i) *The trivial geometry on the universe of M*
- (ii) *An affine geometry of infinite dimension over a finite field*
- (iii) *A projective geometry of infinite dimension over a finite field*

These cases essentially correspond to the cases when M is a set with no extra structure, when M is an infinite dimensional vector space over a finite field, and when M the projectivisation, in the usual algebraic sense, of an infinite dimensional vector space over a finite field, respectively.

Zil'ber [Zil'ber, b], [Zil'ber, c] and independently Cherlin [G. Cherlin, 1985] and Miller (unpublished), proved that Zil'ber's trichotomy holds for *strictly minimal* sets, or strongly minimal subsets of $\aleph_0$ -categorical, $\aleph_0$ -stable structures with no nontrivial definable equivalence classes. We do not delve further into the work of Cherlin, Harrington, and Lachlan in [G. Cherlin, 1985], where they developed a structure theory for $\aleph_0$ -categorical, $\aleph_0$ -stable theories. However, we note that, with Morley's result that uncountable categoricity implies $\aleph_0$ -stability and the Baldwin–Lachlan result that any uncountably categorical structure with finitely many countable models is totally categorical, this analysis complements the previous section in understanding the variation in uncountably categorical structures. It also provides an important backdrop for the Hrushovski construction discussed in section 4.2.

3.3. The Computability Theory Perspective. In this subsection, we discuss a paper of Harrington [Harrington, 1974] in which he studied the effects of uncountable categoricity from the perspective of computability theory. This paper was an

early example of a fruitful interaction between model theory and computability theory.

The results in this section were in fact discovered independently around the same time by mathematicians working in the Soviet Union. This was fairly common in computability theory at the time, as the Soviet and Western schools developed in parallel with little interaction.

A theory is *decidable* if there is an algorithm which, when given a sentence in the language of the theory, outputs whether or not it is a consequence of the theory. Given a structure M over a language L , we can define a theory called the *elementary diagram* of M in $L \cup \text{dom}(M)$ consisting of all sentences true in M . A countable model is *decidably presentable* if there is an enumeration of it with respect to which its elementary diagram is decidable. For a deeper introduction to computable model theory, see [Harizanov, 1998].

In this context, Harrington's result, due independently to Khisamiev [Khisamiev, 1974], is that, for any $\aleph_1$ - but not $\aleph_0$ -categorical theory T , every countable model of T is decidably presentable. Thus, not only does uncountable categoricity imply significant model-theoretic structure, as seen in the previous two subsections, but also considerable computability-theoretic power. We will analyse this relationship in more detail in section 5.1.

Harrington's theorem on decidable, uncountably categorical theories follows from the following theorem, in the same paper. It was also independently proved by Goncharov and Nurtazin in the USSR [Goncharov and Nurtazin, 1973].

Theorem 3.3.1. *Let T be a complete decidable theory. Then the following are equivalent:*

- (i) *T has a decidably presentable prime model.*
- (ii) *There is a computable function mapping formulas $\phi(x)$ consistent with T to a computable principal type Γ_ϕ such that $\phi(x) \in \Gamma_\phi$.*

Before we discuss the proof, we show that it indeed gives the result above.

Corollary 3.3.2. *Let T be a complete, decidable, uncountably categorical theory. Then every countable model of T is decidably presentable.*

Proof sketch. By the Baldwin–Lachlan proof, any countable model of a complete, decidable, uncountably categorical theory is a prime model of some other complete, decidable, uncountably categorical theory T' . So it suffices to show (ii) holds for T' .

Let $\phi(x)$ be a formula consistent with T' , and fix a computable order $(\phi_n)_{n < \omega}$ of the formulas in the language of T' . We can computably define a type by at each stage adding ϕ_n if it decreases the rank of the partial type and $\neg\phi_n$ otherwise. Because rank is a computable function, this is an effective method for giving a type, and it can be checked that this type is principal.

□

We are now in a position to sketch a proof of the main theorem. Note that the (i) implies (ii) direction is clear. Thus, it suffices to build a decidably presentable prime model from a computable function f from formulas to principal types. This is done via the classical Henkin construction, which can be found in [Chang and Keisler, 1990]. For the Henkin construction, we need a complete T' extending T in $L \cup C$, where C is the set of new constant symbols. To ensure the model is prime, it suffices to ensure that every n -tuple satisfies a principal type. However, these constraints may be in conflict with one another, so satisfying one may require us to “undo” another. Thus, to satisfy these constraints simultaneously, something more powerful than a simple induction construction is necessary.

Instead Harrington employs a computability theoretic technique called a *finite injury priority argument*, as can be found in [Soare, 1987].

Essentially, he inductively builds the theory T' in $L \cup C$, and, at each stage in the induction, guesses a principal type for each n -tuple. At the n th step, whichever of the n th sentence or its negation does the “least damage” is added, and, for each n -tuple, a principal type is chosen containing all the sentences that the tuple must satisfy according to the initial segment of T' that has been defined. Since only finitely many constraints become broken in any stage, and all the constraints which are broken are “lower priority”, this process eventually gives a complete theory T' with Henkin model M that is prime. Moreover, since “least damage” and finding a principal type containing a sentence are both computable, M is decidable presentable.

4. BUILDING NEW MODELS: THE HRUSHOVSKI CONSTRUCTION

In section 3, we studied the geometry of strongly minimal sets in order to understand the finer structure of countable models of uncountably categorical theories. Cherlin and Miller, and independently Zil’ber, showed that Zil’ber’s trichotomy, Conjecture 3.2.2 above, holds for strongly minimal sets in totally categorical theories. However, in 1993, Hrushovski overturned the Zil’ber conjecture by building an example of a strongly minimal set not sitting within the trichotomy.

Hrushovski’s construction was a very interesting and powerful extension of a classical construction used in model theory, the Fraïssé construction.

4.1. The Fraïssé Construction. In 1953, Roland Fraïssé published a deeply influential paper [Fraïssé, 1954] in which he demonstrated that the rational numbers as a linear order can be realised as a “limit” of finite linear orders. This construction has since been applied to many other collections of structures and has given rise to a rich theory of Fraïssé limits.

Theorem 4.1.1 (Fraïssé). *Let L be a countable language and let $\mathcal{K}$ be a non-empty finite or countable collection of finitely-generated L -structures. Suppose in addition $\mathcal{K}$ satisfies:*

- (i) the **Hereditary Property (HP)**: *for any $A \in \mathcal{K}$ and any B a finitely generated substructure of A , $B \in \mathcal{K}$.*
- (ii) the **Joint Embedding Property (JEP)**: *for any two structures A and B in $\mathcal{K}$, there is a $C \in \mathcal{K}$ such that A and B both embed into C .*
- (iii) the **Amalgamation Property (AP)**: *for any structure $D \in \mathcal{K}$ and any A and B in $\mathcal{K}$ with $D \hookrightarrow A, B$, there exists a $C \in \mathcal{K}$ with embeddings $f : A \hookrightarrow C$ and $g : B \hookrightarrow C$ that agree on the embeddings of D into A and B .*

Then there is a unique countable or finite structure M , called the Fraïssé limit, such that:

- (i) *every finite substructure of M is in $\mathcal{K}$ and every isomorphism class of elements of $\mathcal{K}$ appears as a substructure of M*
- (ii) *M is ultrahomogeneous; i.e., any partial automorphism between finitely generated substructures of M extends to a full automorphism*

Let us consider some examples before we move on to the proof:

Example 4.1.2. (i) The collection of finite linear orders is a Fraïssé class with Fraïssé limit $\mathbb{Q}$.

(ii) The collection of finite graphs is a Fraïssé class with limit the random graph.

- (iii) For every n , the collection of finite n clique-free graphs is a Fraïssé class with Fraïssé limit the random n clique-free graph.

Armed with these examples, we are now in a good place to prove Fraïssé's Theorem.

Sketch of proof of Fraïssé's Theorem. Let $(K_i)_{i < \omega}$ be an enumeration of the isomorphism types of elements of K . Using joint embedding and amalgamation, one can build $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ inductively so that

- (i) $K_n \hookrightarrow M_n$
- (ii) If $i, j \leq n$ with $\alpha_{i,j} : K_i \hookrightarrow K_j$ and $\beta_i : K_i \hookrightarrow M_n$ then there is a $\gamma_j : K_j \hookrightarrow M_{n+1}$ extending β_i .

The required Fraïssé limit will be $M = \bigcup_{i < \omega} M_i$.

The second requirement says that for any K_i appearing in our limit model M , and any K_j extending K_i , K_j will appear in M extending K_i in the appropriate way. This will allow us to prove the ultrahomogeneity of M .

By construction, it is clear that every structure in $\mathcal{K}$ appears as a finitely generated substructure of M . Now, let N be a finitely generated substructure of M . Then N appears in some M_n . But, by construction, for each $M_n \in \mathcal{K}$, any finitely generated substructure of any M_n is in $\mathcal{K}$, as required. $\square$

4.2. Hrushovski Construction. Expanding upon the framework of Fraïssé's construction, Hrushovski developed a technical and powerful method of building new models [Hrushovski, 1993]. While Fraïssé's work had been modified and revised, the Hrushovski construction goes much further than earlier adaptations in its technical depth and its far-reaching applications, and is now a celebrated and widely-applied tool in its own right.

In this note, we will focus on Hrushovski's hypergraph construction. He also gave a related construction for graphs, which solved a different open problem.

To begin, fix a language L with a single ternary relation R . For ease of communication, we will use the graph-theoretic terminology of "vertices" and "hyperedges" to refer to the elements of an L -structure and triples of elements in an L -structure related by R , respectively.

Definition 4.2.1. Let $A \subseteq B$ be finite subsets of an L -structure M . Write $n(A)$ for $|A|$ and $r(A)$ for the number of triples of elements of A on which R holds. Then:

- (i) The *predimension* of A in B is given by $d_0(A) = n(A) - r(A)$
- (ii) The *dimension* of A in $B \subseteq M$ is given by $d(A, B) = \min\{d_0(C) : A \subseteq C \subseteq B \text{ and } C \text{ is finite}\}$
- (iii) A is *self-sufficient* in $B \subseteq M$ if $d_0(A, B) = d(A, B)$. We write this as $A \leq B$

Definition 4.2.2. Let A, B be finite subsets of M . A is *simply algebraic over B* if A and B are disjoint, $d_0(B) = d_0(A \cup B)$, and for any proper nontrivial subset $A_0 \subsetneq A$, $d_0(B) < d_0(A_0 \cup B)$.

A is *minimally simply algebraic over B* if, in addition, A is not simply algebraic over B_0 for any proper, nonempty subset $B_0 \subsetneq B$.

So A is simply algebraic over B if, as you add the points of A to B , the predimension of the resulting structure is larger than the predimension of B until you have added all points of A . A quick argument shows that A is minimally simply algebraic over B just means that every element of B is related by R to some element of A .

Finally, our eventual model M will depend on a parameter μ , which is a function from the quantifier-free type of (A, B) with A minimally simply algebraic over B to

the natural numbers. The function μ is required to satisfy the additional constraint that $\mu(A, B) \geq d_0(B)$.

We define $\mathcal{K}_\mu$ to be the collection of finite L -structures A such that:

- (i) $\emptyset \leq A$
- (ii) If $(A_i)_{i \leq n}$ are disjoint subsets of A , all minimally simply algebraic over $B \subseteq A$, and such that the quantifier free type of (A_i, B) is independent of i , then $n \leq \mu(A_1, B)$

The first condition requires that any induced substructure of A must have non-negative predimension; i.e., it will have more vertices than hyperedges. The second condition provides a cap on the number of pairs (A', B) with A' minimally simply algebraic over B which have the same quantifier free type.

Letting X be a collection of n unrelated vertices shows that $\mathcal{K}_\mu$ is nonempty for any μ .

Now that we have this nonempty collection of finite structures, it suffices to show they satisfy amalgamation so that we can apply the proof of Fraïssé's theorem to obtain a countable model M_μ .

Proposition 4.2.3. *Suppose $A, A_1, A_2 \in \mathcal{K}$ satisfy that $A \leq A_1$ and $A \leq A_2$. Then there is some $B \in \mathcal{K}$ and embeddings $f : A_1 \rightarrow B$ and $g : A_2 \rightarrow B$ such that $fA_1 \leq B$, $gA_2 \leq B$, and $f \upharpoonright A = g \upharpoonright A$.*

The proof of this proposition is quite technical and requires much casework involving the possible combinatorial structures that might arise.

Thus, Fraïssé's proof builds us a countable L structure M_μ such that:

- (i) every finite substructure of M is in $\mathcal{K}_\mu$
- (ii) for every $A \leq M$ and $A \leq B \in \mathcal{K}$, there is an embedding $f : B \rightarrow M$ such that $fB \leq M$ and $f \upharpoonright A = \text{id}$

Note if M_0 and M_1 are two countable L -structures satisfying the above conditions, and $f : A_0 \rightarrow A_1$ is an embedding, then a back-and-forth construction extends f to an isomorphism between M_0 and M_1 . Thus, M_μ is the unique model satisfying the conditions above.

Furthermore, Hrushovski shows that the homogeneity property (ii) above is equivalent to a first order statement, so any countable elementary extension of M_μ also satisfies (i) and (ii). Hence, any countable elementary extension of M_μ is isomorphic to M_μ , so there are only countably many types realised in countable elementary extensions of M_μ . Thus, there is a saturated countable elementary extension of M_μ , to which M_μ is isomorphic.

Moreover, Hrushovski shows M_μ is strongly minimal and does not interpret an infinite group. Demonstrating the latter condition is particularly involved. Since M_μ does not interpret an infinite group, its geometry cannot be affine or projective. Since it is clear that it is not trivial, it must live outside the Zil'ber trichotomy.

It is reasonable to wonder whether some sort of finite classification of geometries still exists, notwithstanding the Hrushovski construction. However, in the same paper, Hrushovski used a variant of his construction, designed so that the model-theoretic data could be "read off" from the geometric information, to show that there are in fact continuum many different geometries. Thus there is no hope of salvaging Zil'ber's classification program in its full generality.

5. LOOKING AHEAD

5.1. Computable Spectra. By the Baldwin–Lachlan theorem, any $\aleph_1$ -categorical theory which is not $\aleph_0$ -categorical has countably many countable models which can be arranged in a chain of order type $\omega + 1$: $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \dots \prec \mathcal{M}_\omega$. This raises the

natural computability-theoretic question of which of these models are *computable* in a precise sense defined below.

Definition 5.1.1. Given a structure M for a language L , define the *quantifier-free diagram* of M to be the set of all quantifier free sentences in $L \cup M$ which hold in the model M where every element is named by a constant.

A countable model M is *computably presentable*, or, by abuse of terminology *computable*, if there is some enumeration $M = (m_i)_{i < \omega}$ for which the quantifier-free diagram of M is decidable.

Comparing to the notion of *decidably presentable* from section 3.3, we see that being computably presentable is a weaker property in general.

Definition 5.1.2. The *Spectrum of Recursive Models* of an $\aleph_1$ - but not $\aleph_0$ -categorical theory T , is given by

$$\text{SRM}(T) = \{i : \mathcal{M}_i \text{ is computable.}\}$$

with the order given by the Baldwin–Lachlan theorem.

Note that we use the terminology spectrum of *recursive* and not *computable* models. This is for historic reasons, as the adjective “recursive” used to be used more frequently in place of its synonym “computable”. This convention is standard, but by no means universal.

Now we can make the question at the beginning of the section more precise: which subsets of $\omega + 1$ appear as $\text{SRM}(T)$ for some T ? For example, Harrington’s theorem (Theorem 3.3.2) shows that if T is decidable, then $\text{SRM}(T) = \omega + 1$. It is also straightforward to obtain a theory T for which $\text{SRM}(T) = \emptyset$. The following examples give the next most complicated examples of spectra one can build.

Example 5.1.3 ($\text{SRM}(T_n) = \{0, \dots, n\}$). For every natural number n , we will find some T_n such that $\text{SRM}(T_n) = \{0, \dots, n\}$.

So fix $n < \omega$ and a computably enumerable, noncomputable set K . Let K_s denote the subset of K defined at the s th stage in the enumeration of K and let $L = \{R_i : i < \omega\} \cup \{c_j : j < \omega\}$, where each R_i is an $n + 1$ -ary relation symbol and each c_j is a constant symbol. Define a model M_n with universe ω such that c_j names j and $R_i(\bar{x})$, $\bar{x} = (x_0, \dots, x_n)$, if and only if $x_\ell \neq x_k$ for $\ell \neq k$ and $i \in K_{\min(\bar{x})}$. Finally, let $T_n = \text{Th}(M_n)$.

First note that, if A and B are two models of T of size $\aleph_1$, the map $c_j^A \mapsto c_j^B$ extends to an isomorphism. Moreover, T is not countably categorical, as there is a model M_k with k elements not named by constants, for each $k \leq \omega$.

Given R_i and an $n + 1$ -tuple $\bar{x}$, we can determine if $R_i(\bar{x})$ using the following algorithm:

For $m < \omega$, if, for some j , $x_j = c_m$, and if $i \in K_m$, then $R_i(\bar{x})$ holds. If for some j , $x_j = c_m$, and if $i \notin K_m$, then $\neg R_i(\bar{x})$. If no $x_j = c_m$, then set $m = m + 1$ and proceed.

This algorithm will compute R_i for any tuple with at least one element named by a constant. So M_k is computable for $k \leq n$. On the other hand, if $i \in K$, then M proves that if each $x_j \geq m$, then $R_i(\bar{x})$ holds, and if $i \notin K$, then M proves that $\forall \bar{x} \neg R_i(\bar{x})$. Thus, if $\bar{x}$ doesn’t contain any constants (which is only possible when $k > n$ because we stipulate that we cannot repeat coordinates), $i \in K$ if and only if $R_i(\bar{x})$. Thus M_k is not computable for $k > n$.

One can also obtain $\text{SRM}(T) = \omega$, as the following example demonstrates.

Example 5.1.4 ($\text{SRM}(T) = \omega$). Fix a set $S = \{i : \forall k \exists \ell \phi(i, k, \ell)\}$ which cannot be defined by a $\exists \forall$ formula and let $L = \{R_{s,i} : s, i < \omega\} \cup \{c_j : j < \omega\}$, where each

$R_{i,s}$ is an i -ary relation and each c_j is a constant symbol. Define a model M with universe ω such that c_j names j and $R_{s,i}(\bar{x})$ if and only if $x_\ell \neq x_k$ for $\ell \neq k$ and $R_{s,i}(\bar{x})$ holds iff $\forall k \leq s \exists \ell \leq \min \bar{x} \phi(i, k, \ell)$. Let $T = Th(M)$.

As above, T is uncountably categorical and its countable models M_k , $k \leq \omega$ correspond to adding k elements not named by constant symbols.

To show M_ω is not computable, note that $i \in S$ iff

$$\exists \bar{x} \left(\bigwedge_{i \neq j} x_i \neq x_j \wedge \forall s M_\omega \models R_{i,s}(\bar{x}) \right).$$

If M_ω were computable, we could thus define S by a $\exists \forall$ formula, a contradiction.

On the other hand, if M_n is a finite-dimensional countable model, it suffices to show that one can computably tell whether $R_{i,s}(\bar{x})$, given $\bar{x}$ for each i, s . If $\bar{x}$ contains an element named by a constant, this is clear, as one only needs to test a finite initial segment of S . However, there are only finitely many $i < n$ for which an i -tuple $\bar{x}$ of distinct elements may not contain a constant symbol. For each such i , we just need to build into our program whether or not that $i \in S$, and if not, after which s , $\forall k < s \exists \ell \phi(i, k, \ell)$ fails. Since this is finite information, M_n is computable.

Note that in the previous two examples, we used an infinite language. Moreover, in all of the examples so far, the spectra were intervals. It was an open question as to whether one could have non-interval spectra or whether one could realise the spectra in the previous examples in a finite language.

In 2010, using a variant of the Hrushovski construction, Uri Andrews was able to do both of these things. Prior to Andrews' work, only one nontrivial (i.e., not $\emptyset$ or $\omega + 1$) example of a computable spectrum had been given in a finite language. Herwig, Lempp, and Ziegler showed that there is a theory T in a finite language such that the only computable model is the prime model [B. Herwig, 1999]. Moreover, no examples had been given of a spectrum that was not an interval.

As in Hrushovski's original construction, let $L = \{R\}$, where R is a ternary relation. Recall that we defined a dimension function d in Definition 4.2.2.

Definition 5.1.5. Let $A \subseteq B$, A finite, be L -structures. Then $f_B(A) = \min\{|C| : A \subseteq C \subseteq B, C \text{ finite, and } d(C) < |A|\}$. We follow the convention that $\min \emptyset = \infty$.

Notice that this in some way measures how well-connected A is, within B ; i.e., $f_B(A)$ counts the number of vertices one needs to add to A so that, on average, one adds more hyperedges than vertices.

We now modify μ , from the Hrushovski construction, to allow for dependence on this new function.

Definition 5.1.6. Let $\mu(A, B, n)$ be a function on the quantifier-free types of two finite L -structures A and B with A minimally simply algebraic over B , and $n \in \omega \cup \{\infty\}$. We suppose in addition that $\mu(A, B, n) = \mu(A, B, \infty)$ for all but finitely many n and that $\mu(A, B, n) \geq d(A)$.

We can now define the class $\mathcal{K}_\mu$ very similarly to section 4.2.

Definition 5.1.7. Let $\mathcal{K}_\mu$ be the collection of finite L -structures A satisfying:

- (i) $d(B) \geq 0$ for every $B \subseteq A$
- (ii) If $(A_i)_{i \leq n}$ are disjoint subsets of A , all minimally simply algebraic over $B \subseteq A$, and such that the quantifier free type of (A_i, B) is independent of i , then $n \leq \mu(A_1, B, f_A(B))$

Note that this last condition implies that it is possible that $B \subseteq A \in \mathcal{C}$, but $B \notin \mathcal{C}$. However, we still have that if $B \leq A \in \mathcal{C}$, then $B \in \mathcal{C}$. This changes some of the arguments in Hrushovski's paper slightly, but not materially. In particular,

$\mathcal{K}_\mu$ still satisfies amalgamation, so we can again apply Fraïssé's construction to obtain a generic countable structure M satisfying:

- (i) For $A \leq M$, $A \in \mathcal{C}$
- (ii) Suppose $A \leq M$ and $B \leq A$. Then there is an embedding $g : B \hookrightarrow M$ such that $g \upharpoonright_A$ is the identity.

As before, these can be reformulated in a first order way, which one can use to show that M is saturated and strongly minimal. Thus the theory of M is uncountably categorical.

Now fix $m < \omega$, set K to be the halting problem, and define K_s to be the subset of K enumerated by stage s . We can define a computable list of all the quantifier-free types of a pair (A, B) , where $|A| = m + 1$ and B is minimally simply algebraic over A . Write $\Delta_i(A, B)$ for the i th type in this enumeration. Define $\mu(A, B, n)$ as follows:

$$\mu(A, B, n) = \begin{cases} |A| + 1 & \text{if } |A| \neq m + 1 \\ |A| + 1 & \text{if } \Delta_i(A, B) \text{ and } i \in K_n \\ |A| + 2 & \text{if } \Delta_i(A, B) \text{ and } i \notin K_n \end{cases}$$

This gives rise to a countable model M_μ and uncountably categorical but not countably categorical theory $T = Th(M_\mu)$.

We now sketch an argument that $SRM(T) = \{0, \dots, m\}$.

First, take a countable model N of dimension $k > m$. Then, for any $m + 1$ -tuple $\bar{x}$ of independent elements, we have that $i \in K$ if and only if $\neg \exists^{m+3} \bar{y} \Delta_i(\bar{x}, \bar{y})$ by construction. Thus N cannot be recursive.

We now build M inductively. Start by setting M_0 to be k elements with no relations between them. At stage n , consider the first n possible simply algebraic extensions over subsets of M_{n-1} . For each simply algebraic extension, if its free join is in $\mathcal{K}_\mu$ (which Andrews argues can be checked computably), take the free join. Let M_n be the result of including as many free joins as are in $\mathcal{K}_\mu$. $\bigcup_{n < \omega} M_n$ will be the required computable model of dimension k .

Hence, we have found a theory T in a countable language with $SRM(T) = \{0, \dots, m\}$.

A similar, but slightly more complicated, idea gives a theory T in a countable language with $SRM(T) = \omega$.

In [Andrews, 2011], Andrews uses a different modification of the Hrushovski construction that allows it to work for infinite languages, to give an example, in an infinite language, of a theory T such that $SRM(T) = \{0, \omega\}$. This was the first example of a non-interval spectrum.

This is still an active area of research and much progress has been made, both extending Andrews' work and in other directions. Nonetheless, there is still much to be done in this. For example, it is unknown whether there can be an infinite, co-infinite spectrum, and, even more generally, there are no results prohibiting certain subsets from appearing as computable spectra. It would be interesting to find deeper connections between properties of T and possible $SRM(T)$. However, as Andrews' work illustrates, even getting very simple examples of spectra required heavy model-theoretic machinery, which hints at the difficulty and depth of this problem.

5.2. Forcing in the Model Theory Context. Very shortly after Paul Cohen developed forcing to prove the independence of the Continuum Hypothesis, Abraham Robinson translated it into the world of model theory, creating *model-theoretic forcing* [Robinson, 1971]. Continuing in the vein of using forcing-like ideas to build structures, Angus Macintyre used this model-theoretic forcing to better understand the recursive properties of finitely generated groups.

What Macintyre showed is that, if a finitely generated group H can be embedded into every nontrivial algebraically closed group, then it must be that H is computably presentable with a solvable word problem. The converse was already known to B.H. Neumann [Neumann, 1973], so this is a characterisation.

The proof uses a model-theoretic forcing argument, so we lay out the bare bones of that technique here; for a more complete overview, see [Robinson, 1971] or [J. Barwise, 1970].

Fix a (not necessarily complete) theory T in a language L , and a collection A of constant symbols not in L . A *condition* is a finite collection of atomic or negated atomic sentences in $L \cup A$ that is consistent with T . Write $C = C(T, A)$ for the collection of conditions.

We now define a *forcing relation* $P \Vdash \phi$, read “ P forces ϕ ” where P is a condition and ϕ is a sentence in $L \cup A$. The definition is by induction on the complexity of ϕ .

Definition 5.2.1. For a condition $P \in C(T, A)$ and a sentence ϕ in $L \cup A$, define $P \Vdash \phi$ as follows:

- (i) If ϕ is atomic, $P \Vdash \phi$ exactly when $\phi \in P$
- (ii) If $\phi = \psi_1 \vee \psi_2$, $P \Vdash \phi$ exactly when $P \Vdash \psi_1$ or $P \Vdash \psi_2$
- (iii) If $\phi = \exists x \psi(x)$, $P \Vdash \phi$ exactly when there is some term t such that $P \Vdash \psi(t)$
- (iv) If $\phi = \neg \psi$, $P \Vdash \phi$ exactly when there is no condition $Q \supseteq P$ such that $Q \Vdash \psi$

It is not hard to see that one can have $P \Vdash \neg \neg \phi$ but not $P \Vdash \phi$. Thus we define “ P weakly forces ϕ ”, written $P \Vdash^* \phi$ if $P \Vdash \neg \neg \phi$. It follows similarly to the proof in set theoretic forcing that the set of sentences weakly forced by P is consistent and closed under logical consequence, and that if $P \subseteq Q \in C$, and $P \Vdash \phi$, then $Q \Vdash \phi$.

A structure M is T -generic if both the following conditions hold:

- (i) every finite collection of atomic and negated atomic sentences true in M is consistent with T .
- (ii) For any sentence ϕ in $L \cup A$, $M \models \phi$ iff there is some finite P in the quantifier-free diagram of M such that $P \Vdash \phi$.

As in set theoretic forcing, for a countable language L and any given $P \in C(T, A)$, there is a countable T -generic model M with $M \models P$.

Before we can state the main result, recall that the *Turing degree* of a set $X \subseteq \omega$ gives a measure of the relative computability of X . In particular, if $\deg(X) \geq \deg(Y)$, then one can write an algorithm for determining membership in Y , assuming knowledge of X .

Theorem 5.2.2. *Let T be a theory and T_0 a set of axioms for T . Let Δ_m be a quantifier-free type in k_m free variables. If, for every m , it is not the case that $\deg(\Delta_m) \leq \deg(T_0)$, then there is a T -generic structure omitting each Δ_m .*

As a corollary of the Theorem, we get the result mentioned at the beginning of this section concerning when a finitely generated group admits a computable presentation with a solvable word problem.

Corollary 5.2.3. *Let H be a finitely generated group and suppose that H can be embedded in any nontrivial algebraically closed group. Then H is computably presentable with a solvable word problem.*

Proof. Set T to be the theory of groups and T_0 the standard set of axioms for T . As T_0 is finite, it is certainly computable.

Let H satisfy the hypotheses and let $h_0, \dots, h_n$ be generators for H . Let

$$\Delta = \{\phi(v_0, \dots, v_n) \mid \phi \text{ is an atomic or negated atomic formula and } H \models \phi(h_0, \dots, h_n)\}.$$

Then clearly, if Δ is computable, then H has a computable presentation with a solvable word problem.

Suppose Δ is not computable. Then, by the previous theorem, there is a T -generic G such that G omits Δ . However, it was previously known that T -generic groups are exactly the nontrivial algebraically closed groups, so G is algebraically closed. Moreover, since G omits Δ , H cannot possibly embed in G . Thus Δ is computable, as required. $\square$

The proof of the theorem uses model theoretic forcing and a Henkin-like argument to build a model satisfying the constraints. The hypothesis on the relative computability of T_0 and the Δ_m 's essentially guarantees that the set of conditions which “disagree” with some Δ_m is dense, in the sense of forcing [J. Barwise, 1970].

Macintyre’s result illustrates the utility of model theoretic forcing in understanding algebraic structures, and the interactions of computability theory with those structures. While Macintyre’s paper is from 1972, the interactions between these ideas and the various problems that have been discussed seem to not have been fully explored in the intervening decades.

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